ITAMO 1985

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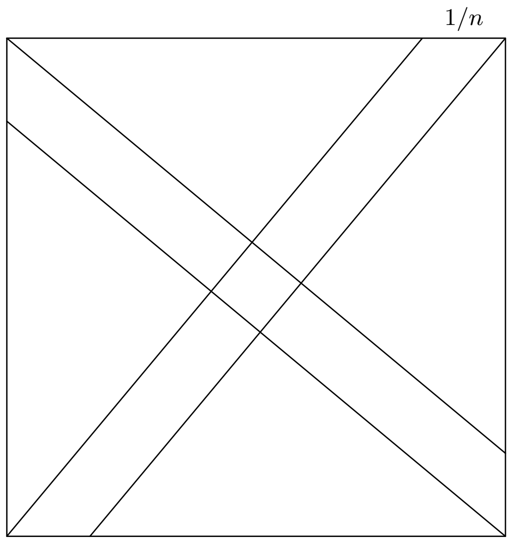
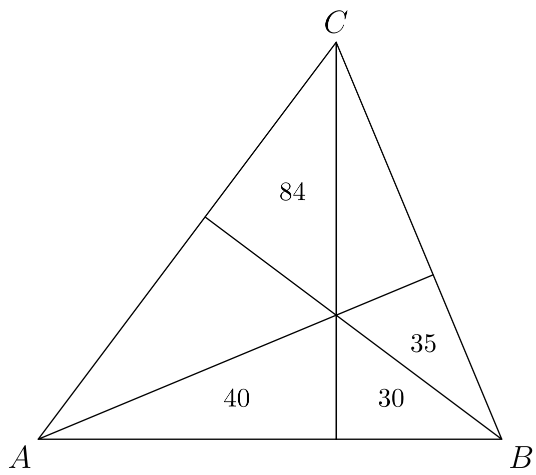
–	In the 1st ITAMO 1985, they used the AIME 1985 problems.
1	Let $x_1 = 97$, and for $n > 1$ let $x_n = \frac{n}{x_{n-1}}$. Calculate the product $x_1 x_2 \cdots x_8$.
2	When a right triangle is rotated about one leg, the volume of the cone produced is 800π cm^3. When the triangle is rotated about the other leg, the volume of the cone produced is 1920π cm^3. What is the length (in cm) of the hypotenuse of the triangle?
3	Find c if a, b, and c are positive integers which satisfy $c = (a + bi)^3 - 107i$, where $i^2 = -1$.
4	A small square is constructed inside a square of area 1 by dividing each side of the unit square into n equal parts, and then connecting the vertices to the division points closest to the opposite vertices. Find the value of n if the the area of the small square is exactly 1/1985.



5	A sequence of integers $a_1, a_2, a_3, \ldots$ is chosen so that $a_n = a_{n-1} - a_{n-2}$ for each $n \geq 3$. What is the sum of the first 2001 terms of this sequence if the sum of the first 1492 terms is 1985, and the sum of the first 1985 terms is 1492?
6	As shown in the figure, triangle ABC is divided into six smaller triangles by lines drawn from

the vertices through a common interior point. The areas of four of these triangles are as indicated. Find the area of triangle ABC.



7 Assume that a, b, c, and d are positive integers such that $a^5 = b^4$, $c^3 = d^2$, and $c - a = 19$. Determine $d - b$.

8 The sum of the following seven numbers is exactly 19:

$$a_1 = 2.56, \quad a_2 = 2.61, \quad a_3 = 2.65, \quad a_4 = 2.71,$$

$$a_5 = 2.79, \quad a_6 = 2.82, \quad a_7 = 2.86.$$

It is desired to replace each a_i by an integer approximation A_i, $1 \le i \le 7$, so that the sum of the A_i's is also 19 and so that M, the maximum of the "errors" $|A_i - a_i|$, is as small as possible. For this minimum M, what is $100M$?

9 In a circle, parallel chords of lengths 2, 3, and 4 determine central angles of α, β, and $\alpha + \beta$ radians, respectively, where $\alpha + \beta < \pi$. If $\cos\alpha$, which is a positive rational number, is expressed as a fraction in lowest terms, what is the sum of its numerator and denominator?

10 How many of the first 1000 positive integers can be expressed in the form

$$\lfloor 2x \rfloor + \lfloor 4x \rfloor + \lfloor 6x \rfloor + \lfloor 8x \rfloor,$$

where x is a real number, and $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z?

11 An ellipse has foci at $(9, 20)$ and $(49, 55)$ in the xy-plane and is tangent to the x-axis. What is the length of its major axis?

12 Let A, B, C, and D be the vertices of a regular tetrahedron, each of whose edges measures 1 meter. A bug, starting from vertex A, observes the following rule: at each vertex it chooses one of the three edges meeting at that vertex, each edge being equally likely to be chosen, and crawls along that edge to the vertex at its opposite end. Let $p = n/729$ be the probability that the bug is at vertex A when it has crawled exactly 7 meters. Find the value of n.

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13 The numbers in the sequence 101, 104, 109, 116, $\dots$ are of the form $a_n = 100+n^2$, where $n = 1$, 2, 3, $\dots$. For each n, let d_n be the greatest common divisor of a_n and a_{n+1}. Find the maximum value of d_n as n ranges through the positive integers.

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15 In a tournament each player played exactly one game against each of the other players. In each game the winner was awarded 1 point, the loser got 0 points, and each of the two players earned 1/2 point if the game was a tie. After the completion of the tournament, it was found that exactly half of the points earned by each player were earned against the ten players with the least number of points. (In particular, each of the ten lowest scoring players earned half of her/his points against the other nine of the ten). What was the total number of players in the tournament?

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15 Three 12 cm $\times$ 12 cm squares are each cut into two pieces A and B, as shown in the first figure below, by joining the midpoints of two adjacent sides. These six pieces are then attached to a regular hexagon, as shown in the second figure, so as to fold into a polyhedron. What is the volume (in cm^3) of this polyhedron?