Czech And Slovak Mathematical Olympiad, Round III, Category A 2017

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by parmenides51

1 There are 100 diamonds on the pile, 50 of which are genuine and 50 false. We invited a peculiar expert who alone can recognize which are which. Every time we show him some three diamonds, he would pick two and tell (truthfully) how many of them are genuine . Decide whether we can surely detect all genuine diamonds regardless how the expert chooses the pairs to be considered.

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2 Find all pairs of real numbers k, l such that inequality $ka^2 + lb^2 > c^2$ applies to the lengths of sides a, b, c of any triangle.

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3 Find all functions $f : R \to R$ such that for all real numbers x, y holds $f(y - xy) = f(x)y + (x - 1)^2 f(y)$

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4 For each sequence of n zeros and n units, we assign a number that is a number sections of the same digits in it. (For example, sequence 00111001 has 4 such sections $00, 111, 00, 1$.) For a given n we sum up all the numbers assigned to each such sequence. Prove that the sum total is equal to $(n + 1)\binom{2n}{n}$

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5 Given is the acute triangle ABC with the intersection of altitudes H. The angle bisector of angle BHC intersects side BC at point D. Mark E and F the symmetrics of the point D wrt lines AB and AC. Prove that the circle circumscribed around the triangle AEF passes through the midpoint of the arc BAC

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6 Given is a nonzero integer k.
Prove that equation $k = \frac{x^2-xy+2y^2}{x+y}$ has an odd number of ordered integer pairs (x, y) just when k is divisible by seven.
