Kosovo National Mathematical Olympiad 2020

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–	Grade 9
1	Compare the following two numbers: $2^{2^{2^{2^2}}}$ and $3^{3^{3^3}}$.
2	A natural number n is written on the board. Ben plays a game as follows: in every step, he deletes the number written on the board, and writes either the number which is three greater or two less than the number he has deleted. Is it possible that for every value of n, at some time, he will get to the number 2020?
3	Let $\triangle ABC$ be a triangle. Let O be the circumcenter of triangle $\triangle ABC$ and P a variable point in line segment BC. The circle with center P and radius PA intersects the circumcircle of triangle $\triangle ABC$ again at another point R and RP intersects the circumcircle of triangle $\triangle ABC$ again at another point Q. Show that points A, O, P and Q are concyclic.
4	Let p and q be prime numbers. Show that p^2+q^2+2020 is composite.
–	Grade 10
1	Let $x \in \mathbb{R}$. What is the maximum value of the following expression: $\sqrt{x-2018}+\sqrt{2020-x}$?
2	Ana baked 15 pasties. She placed them on a round plate in a circular way: 7 with cabbage, 7 with meat and 1 with cherries in that exact order and put the plate into a microwave. She doesnt know how the plate has been rotated in the microwave. She wants to eat a pasty with cherries. Is it possible for Ana, by trying no more than three pasties, to find exactly where the pasty with cherries is?
3	Find all prime numbers p such that 3^p+5^p-1 is a prime number.
4	Let B' and C' be points in the circumcircle of triangle $\triangle ABC$ such that $AB = AB'$ and $AC = AC'$. Let E and F be the foot of altitudes from B and C to AC and AB, respectively. Show that $B'E$ and $C'F$ intersect on the circumcircle of triangle $\triangle ABC$.
–	Grade 11
1	Some positive integers, sum of which is 23, are written in sequential form. Neither one of the terms nor the sum of some consecutive terms in the sequence is equal to 3.

a) Is it possible that the sequence contains exactly 11 terms?
b)Is it possible that the sequence contains exactly 12 terms?

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2 Find all positive integers x, y such that $2^x + 5^y + 2$ is a perfect square.

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3 Let a and b be real numbers such that $a+b = \log_2(\log_2 3)$. What is the minimum value of $2^a + 3^b$?

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4 Let $\triangle ABC$ be a triangle and ω its circumcircle. The exterior angle bisector of $\angle BAC$ intersects ω at point D. Let X be the foot of the altitude from C to AD and let F be the intersection of the internal angle bisector of $\angle BAC$ and BC. Show that BX bisects segment AF.

– Grade 12

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1 Two players, Agon and Besa, choose a number from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$, in turns, until no number is left. Then, each player sums all the numbers that he has chosen. We say that a player wins if the sum of his chosen numbers is a prime and the sum of the numbers that his opponent has chosen is composite. In the contrary, the game ends in a draw. Agon starts first. Does there exist a winning strategy for any of the players?

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2 Let $a_1, a_2, ..., a_n$ be integers such that $a_1^{20} + a_2^{20} + ... + a_n^{20}$ is divisible by 2020. Show that $a_1^{2020} + a_2^{2020} + ... + a_n^{2020}$ is divisible by 2020.

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3 Let ABC be a triangle with incenter I. The points D and E lie on the segments CA and BC respectively, such that $CD = CE$. Let F be a point on the segment CD. Prove that the quadrilateral $ABEF$ is circumscribable if and only if the quadrilateral $DIEF$ is cyclic.

Proposed by Dorlir Ahmeti, Albania

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4 Let a_0 be a fixed positive integer. We define an infinite sequence of positive integers $\{a_n\}_{n\geq 1}$ in an inductive way as follows: if we are given the terms $a_0, a_1, ...a_{n-1}$, then a_n is the smallest positive integer such that $\sqrt[n]{a_0 \cdot a_1 \cdot ... \cdot a_n}$ is a positive integer. Show that the sequence $\{a_n\}_{n\geq 1}$ is eventually constant.
Note: The sequence $\{a_n\}_{n\geq 1}$ is eventually constant if there exists a positive integer k such that $a_n = c$, for every $n \geq k$.
