Kosovo Team Selection Test 2020

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by dangerousliri, Leartia

1 Find all functions $f:\mathbb{R}\rightarrow\mathbb{R}$ such that, for all real numbers x and y satisfy,

$$f\left(x+yf(x+y)\right)=y^2+f(x)f(y)$$

Proposed by Dorlir Ahmeti, Kosovo

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2 Let p be an odd prime number. Ana and Ben are playing a game with alternate moves as follows: in each move, the player which has the turn choose a number, which was not choosen before by any of the player, from the set $\{1,2,...,2p-3,2p-2\}$. This process continues until no number is left. After the end of the process, each player create the number by taking the product of the choosen numbers and then add 1. We say a player wins if the number that did create is divisible by p, while the number that did create the opponent it is not divisible by p, otherwise we say the game end in a draw. Ana start first move.

Does it exist a strategy for any of the player to win the game?

Proposed by Dorlir Ahmeti, Kosovo

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3 Let $ABCD$ be a cyclic quadrilateral with center O such that BD bisects AC. Suppose that the angle bisector of $\angle ABC$ intersects the angle bisector of $\angle ADC$ at a single point X different than B and D. Prove that the line passing through the circumcenters of triangles XAC and XBD bisects the segment OX.

Proposed by Viktor Ahmeti and Leart Ajvazaj, Kosovo

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4 Prove that for all positive integers m and n the following inequality hold:

$$\pi(m)-\pi(n)\leq\frac{(m-1)\varphi(n)}{n}$$

When does equality hold?

Proposed by Shend Zhjeqi and Dorlir Ahmeti, Kosovo
