Czech And Slovak Mathematical Olympiad, Round III, Category A 1991

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by parmenides51

1 Prove that for any real numbers $p,q,r,\phi,$:

$$\cos^2\phi + q\sin\phi\cos\phi + r\sin^2\phi \geq \frac{1}{2}(p+r-\sqrt{(p-r)^2+q^2})$$

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2 A museum has the shape of a (not necessarily convex) 3n-gon.
Prove that n custodians can be positioned so as to control all of the museums space.

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3 For any permutation p of the set $\{1,2,...,n\}$, let us denote $d(p) = |p(1)-1|+|p(2)-2|+...+|p(n)-n|$.
Let $i(p)$ be the number of inversions of p, i.e. the number of pairs $1 \leq i < j \leq n$ with $p(i) > p(j)$.
Prove that $i(p) \leq d(p)$.

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4 Prove that in all triangles ABC with $\angle A = 2\angle B$ the distance from C to A and to the perpendicular bisector of AB are in the same ratio.

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5 In a group of mathematicians everybody has at least one friend (friendship is a symmetric relation). Show that there is a mathematician all of whose friends have average number of friends not smaller than the average number of friends in the whole group.

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6 The set N is partitioned into three (disjoint) subsets A_1, A_2, A_3.
Prove that at least one of them has the following property: There exists a positive number m such that for any k one can find numbers $a_1 < a_2 < ... < a_k$ in that subset satisfying $a_{j+1} - a_j \leq m$ for $j = 1,...,k-1$.
