Niels Henrik Abels Math Contest (Norwegian Math Olympiad) Final Round 1994

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by parmenides51

1a In a half-ball of radius 3 is inscribed a cylinder with base lying on the base plane of the half-ball, and another such cylinder with equal volume. If the base-radius of the first cylinder is $\sqrt{3}$, what is the base-radius of the other one?

1b Let C be a point on the extension of the diameter AB of a circle. A line through C is tangent to the circle at point N. The bisector of $\angle ACN$ meets the lines AN and BN at P and Q respectively. Prove that $PN = QN$.

2a Find all primes p, q, r and natural numbers n such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1}{n}$.

2b Find all integers x, y, z such that $x^3 + 5y^3 = 9z^3$.

3a Let $x_1, x_2, ..., x_{1994}$ be positive real numbers. Prove that

$$\left(\frac{x_1}{x_2}\right)^{\frac{x_1}{x_2}} \left(\frac{x_2}{x_3}\right)^{\frac{x_2}{x_3}} \cdots \left(\frac{x_{1993}}{x_{1994}}\right)^{\frac{x_{1993}}{x_{1994}}} \geq \left(\frac{x_1}{x_2}\right)^{\frac{x_2}{x_1}} \left(\frac{x_2}{x_3}\right)^{\frac{x_3}{x_2}} \cdots \left(\frac{x_{1993}}{x_{1994}}\right)^{\frac{x_{1994}}{x_{1993}}}$$

3b Prove that there is no function $f : Z \to Z$ such that $f(f(x)) = x + 1$ for all x.

4a In a group of 20 people, each person sends a letter to 10 of the others.
Prove that there are two persons who send a letter to each other.

4b Finitely many cities are connected by one-way roads. For any two cities it is possible to come from one of them to the other (with possible transfers), but not necessarily both ways. Prove that there is a city which can be reached from any other city, and that there is a city from which any other city can be reached.
