Japan MO Finals 2020

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by maple116

1 Find all pairs of positive integers (m,n) such that $\frac{n^2+1}{2m}$ and $\sqrt{2^{n-1}+m+4}$ are both integers.

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2 Triangle ABC satisfies $BC<AB$ and $BC<AC$. Points D,E lie on segments AB,AC respectively, satisfying $BD=CE=BC$. Lines BE and CD meet at point P, circumcircles of triangle ABE and ACD meet at point Q other than A. Prove that lines PQ and BC are perpendicular.

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3 Find all functions $f:\mathbb{Z}^+\to\mathbb{Z}^+$ such that

$$m^2+f(n)^2+(m-f(n))^2\geq f(m)^2+n^2$$

for all pairs of positive integers (m,n).

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4 Let $n\geq 2$ be an integer. $3n$ distinct points are plotted on the circle, where A and B perform the following operation : Firstly, A picks exactly 2 points which haven't been connected yet and connects them by a segment. Secondly, B picks exactly 1 point with no piece and place a piece. Prove that, after consecutive n operations, despite how B acts, A can make the number of segments no less than $\frac{n-1}{6}$, which connect a point with a piece and a point with no piece.

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5 Find all infinite sequences of positive integers $\{a_n\}_{n\geq 1}$ satisfying the following condition : there exists a positive constant c such that $\gcd(a_m+n,a_n+m)>c(m+n)$ holds for all pairs of positive integers (m,n).
