Ukraine Team Selection Test 1999

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–	Day 1
1	A triangle ABC is given. Points E, F, G are arbitrarily selected on the sides AB, BC, CA, respectively, such that $AF \perp EG$ and the quadrilateral $AEFG$ is cyclic. Find the locus of the intersection point of AF and EG.
2	Show that there exist integers j, k, l, m, n greater than 100 such that $j^2 + k^2 + l^2 + m^2 + n^2 = jklmn - 12$.
3	Let m, n be positive integers with $m \leq n$, and let F be a family of m-element subsets of $\{1, 2, ..., n\}$ satisfying $A \cap B \neq \varnothing$ for all $A, B \in F$. Determine the maximum possible number of elements in F.
–	Day 2
4	If $n \in N$ and $0 < x < \frac{\pi}{2n}$, prove the inequality $\frac{\sin 2x}{\sin x} + \frac{\sin 3x}{\sin 2x} + ... + \frac{\sin(n+1)x}{\sin nx} < 2\frac{\cos x}{\sin^2 x}$. .
5	A convex pentagon $ABCDE$ with $DC = DE$ and $\angle DCB = \angle DEA = 90^o$ is given. Let F be a point on the segment AB such that $AF : BF = AE : BC$. Prove that $\angle FCE = \angle ADE$ and $\angle FEC = \angle BDC$.
6	Show that for any $n \in N$ the polynomial $f(x) = (x^2 + x)^{2^n} + 1$ is irreducible over $Z[x]$.
–	Day 3
7	Let $P_1P_2...P_n$ be an oriented closed polygonal line with no three segments passing through a single point. Each point P_i is assinged the angle $180^o - \angle P_{i-1}P_iP_{i+1} \geq 0$ if P_{i+1} lies on the left from the ray $P_{i-1}P_i$, and the angle $-(180^o - \angle P_{i-1}P_iP_{i+1}) < 0$ if P_{i+1} lies on the right. Prove that if the sum of all the assigned angles is a multiple of 720^o, then the number of self-intersections of the polygonal line is odd
8	Find all pairs (x, n) of positive integers for which $x^n + 2^n + 1$ divides $x^{n+1} + 2^{n+1} + 1$.
9	Find all functions $u : R \to R$ for which there is a strictly increasing function $f : R \to R$ such that $f(x + y) = f(x)u(y) + f(y)$ for all $x, y \in R$.

– Day 4

10 For a natural number n, let $w(n)$ denote the number of (positive) prime divisors of n.
Find the smallest positive integer k such that $2^{w(n)} \leq k\sqrt[4]{n}$ for each $n \in N$.

11 Let $ABCDEF$ be a convex hexagon such that $BCEF$ is a parallelogram and ABF an equilateral triangle. Given that $BC = 1, AD = 3, CD + DE = 2$, compute the area of $ABCDEF$

12 In a group of $n \geq 4$ persons, every three who know each other have a common signal. Assume that these signals are not repeated and that there are $m \geq 1$ signals in total. For any set of four persons in which there are three having a common signal, the fourth person has a common signal with at most one of them. Show that there three persons who have a common signal, such that the number of persons having no signal with anyone of them does not exceed $[n+3-\frac{18m}{n}]$