Belarus Team Selection Test 2000

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–	Test 1
1.1	Find the minimal number of cells on a 5×7 board that must be painted so that any cell which is not painted has exactly one neighboring (having a common side) painted cell.
1.2	Let P be a point inside a triangle ABC with $\angle C = 90^o$ such that $AP = AC$, and let M be the midpoint of AB and CH be the altitude. Prove that PM bisects $\angle BPH$ if and only if $\angle A = 60^o$.
1.3	Does there exist a function $f : N \to N$ such that $f(f(n-1)) = f(n+1) - f(n)$ for all $n \geq 2$?
1.4	A closed pentagonal line is inscribed in a sphere of the diameter 1, and has all edges of length ℓ. Prove that $\ell \leq \sin \frac{2\pi}{5}$.
–	Test 2
2.1	All vertices of a convex polyhedron are endpoints of exactly four edges. Find the minimal possible number of triangular faces of the polyhedron.
2.2	Real numbers a, b, c satisfy the equation $$2a^3 - b^3 + 2c^3 - 6a^2b + 3ab^2 - 3ac^2 - 3bc^2 + 6abc = 0$$. If $a < b$, find which of the numbers b, c is larger.
2.3	In the Cartesian plane, two integer points $(a_1, b_1$ and (a_2, b_2) are connected if (a_2, b_2) is one of the points $(-a_1, b_1 \pm 1)$, $(a_1 \pm 1, -b_1)$. Show that there exists an infinite sequence of integer points in which every integer point occurs, and every two consecutive points are connected.
2.4	In a triangle ABC with $AC = b \neq BC = a$, points E, F are taken on the sides AC, BC respectively such that $AE = BF = \frac{ab}{a+b}$. Let M and N be the midpoints of AB and EF respectively, and P be the intersection point of the segment EF with the bisector of $\angle ACB$. Find the ratio of the area of $CPMN$ to that of ABC.
–	Test 3
3.1	In a triangle ABC, let $a = BC, b = AC$ and let m_a, m_b be the corresponding medians. Find all real numbers k for which the equality $m_a + ka = m_b + kb$ implies that $a = b$.

3.2 (a) Prove that $\{n\sqrt{3}\} > \frac{1}{n\sqrt{3}}$ for any positive integer n.
(b) Is there a constant $c > 1$ such that $\{n\sqrt{3}\} > \frac{c}{n\sqrt{3}}$ for all $n \in N$?

3.3 Each edge of a graph with 15 vertices is colored either red or blue in such a way that no three vertices are pairwise connected with edges of the same color. Determine the largest possible number of edges in the graph.

– Test 4

4.1 Find all functions $f, g, h : R \to R$ such that $f(x + y^3) + g(x^3 + y) = h(xy)$ for all $x, y \in R$

4.2 Let ABC be a triangle and M be an interior point. Prove that

$$\min\{MA, MB, MC\} + MA + MB + MC < AB + AC + BC.$$

4.3 Prove that for every real number M there exists an infinite arithmetic progression such that:

- each term is a positive integer and the common difference is not divisible by 10
- the sum of the digits of each term (in decimal representation) exceeds M.

– Test 5

5.1 Let AM and AL be the median and bisector of a triangle ABC ($M, L \in BC$).
If $BC = a, AM = m_a, AL = l_a$, prove the inequalities:
(a) $a \tan \frac{a}{2} \leq 2m_a \leq a \cot \frac{a}{2}$ if $a < \frac{\pi}{2}$ and $a \tan \frac{a}{2} \geq 2m_a \geq a \cot \frac{a}{2}$ if $a > \frac{\pi}{2}$
(b) $2l_a \leq a \cot \frac{a}{2}$.

5.2 Let n, k be positive integers such that n is not divisible by 3 and $k \geq n$. Prove that there exists a positive integer m which is divisible by n and the sum of its digits in decimal representation is k.

5.3 Suppose that every integer has been given one of the colours red, blue, green or yellow. Let x and y be odd integers so that $|x| \neq |y|$. Show that there are two integers of the same colour whose difference has one of the following values: $x, y, x + y$ or $x - y$.

– Test 6

6.1 Find the smallest natural number n for which it is possible to partition the set $M = \{1, 2, ..., 40\}$ into n subsets $M_1, ..., M_n$ so that none of the M_i contains elements a, b, c (not necessarily distinct) with $a + b = c$.

6.2 A positive integer $A_k...A_1A_0$ is called monotonic if $A_k \leq .. \leq A_1 \leq A_0$.
Show that for any $n \in N$ there is a monotonic perfect square with n digits.

6.3 Starting with an arbitrary pair (a,b) of vectors on the plane, we are allowed to perform the operations of the following two types:
(1) To replace (a, b) with $(a + 2kb, b)$ for an arbitrary integer $k \neq 0$;
(2) To replace (a, b) with $(a, b + 2ka)$ for an arbitrary integer $k \neq 0$.
However, we must change the type of operetion in any step.
(a) Is it possible to obtain $((1, 0), (2, 1))$ from $((1, 0), (0, 1))$, if the first operation is of the type (1)?
(b) Find all pairs of vectors that can be obtained from $((1, 0), (0, 1))$ (the type of first operation can be selected arbitrarily).

– Test 7

7.1 For any positive numbers a, b, c, x, y, z, prove the inequality $\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} \geq \frac{(a+b+c)^3}{3(x+y+z)}$

7.2 Given a triangle ABC. The points A, B, C divide the circumcircle Ω of the triangle ABC into three arcs BC, CA, AB. Let X be a variable point on the arc AB, and let O_1 and O_2 be the incenters of the triangles CAX and CBX. Prove that the circumcircle of the triangle XO_1O_2 intersects the circle Ω in a fixed point.

7.3 A game is played by n girls ($n \geq 2$), everybody having a ball. Each of the $\binom{n}{2}$ pairs of players, is an arbitrary order, exchange the balls they have at the moment. The game is called nice **nice** if at the end nobody has her own ball and it is called **tiresome** if at the end everybody has her initial ball. Determine the values of n for which there exists a nice game and those for which there exists a tiresome game.

– Test 8

8.1 The diagonals of a convex quadrilateral $ABCD$ with $AB = AC = BD$ intersect at P, and O and I are the circumcenter and incenter of $\triangle\ ABP$, respectively. Prove that if $O \neq I$ then OI and CD are perpendicular

8.2 Prove that there exists two strictly increasing sequences (a_n) and (b_n) such that $a_n(a_n + 1)$ divides $b_n^2 + 1$ for every natural n.

8.3 Prove that the set of positive integers cannot be partitioned into three nonempty subsets such that, for any two integers x, y taken from two different subsets, the number $x^2 - xy + y^2$ belongs to the third subset.