Austrian-Polish Competition 1979

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–	Individual
1	On sides AB and BC of a square $ABCD$ the respective points E and F have been chosen so that $BE = BF$. Let BN be the altitude in triangle BCE. Prove that $\angle DNF = 90$.
2	Find all polynomials of the form $$P_n(x) = n!x^n + a_{n-1}x^{n-1} + \cdots + a_1x + (-1)^n(n+1)$$ with integer coefficients, having n real roots $x_1, \ldots, x_n$ satisfying $k \leq x_k \leq k+1$ for $k = 1, \ldots, n$.
3	Find all positive integers n such that the inequality $$\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n a_i\right) - \sum_{i=1}^n a_i^3 \geq 6\prod_{i=1}^n a_i$$ holds for any n positive numbers $a_1, \ldots, a_n$.
4	Determine all functions $f : N_0 \to R$ satisfying $f(x+y) + f(x-y) = f(3x)$ for all x, y.
5	The circumcenter and incenter of a given tetrahedron coincide. Prove that all its faces are congruent.
6	A positive integer n and a real number a are given. Find all n-tuples $(x_1, ..., x_n)$ of real numbers that satisfy the system of equations $\sum_{i=1}^n x_i^k = a^k$ for $k = 1, 2, ..., n$.
–	Team
7	Let n and m be fixed positive integers. The hexagon $ABCDEF$ with vertices $A = (0,0)$, $B = (n,0)$, $C = (n,m)$, $D = (n-1,m)$, $E = (n-1,1)$, $F = (0,1)$ has been partitioned into $n+m-1$ unit squares. Find the number of paths from A to C along grid lines, passing through every grid node at most once.
8	Let A, B, C, D be four points in space, and M and N be the midpoints of AC and BD, respectively. Show that $$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4MN^2$$

9 Find the greatest power of 2 that divides $a_n = [(3+\sqrt{11})^{2n+1}]$, where n is a given positive integer.
