Harvard-MIT Mathematics Tournament 2020

www.artofproblemsolving.com/community/c1074888

by parmenides51, mcdonalds106_7

– Algebra and Number Theory

---

1 Let $P(x) = x^3 + x^2 - r^2x - 2020$ be a polynomial with roots r, s, t. What is $P(1)$?

Proposed by James Lin.

---

2 Find the unique pair of positive integers (a, b) with $a < b$ for which

$$\frac{2020 - a}{a} \cdot \frac{2020 - b}{b} = 2.$$

Proposed by James Lin.

---

3 Let $a = 256$. Find the unique real number $x > a^2$ such that

$$\log_a \log_a \log_a x = \log_{a^2} \log_{a^2} \log_{a^2} x.$$

Proposed by James Lin.

---

4 For positive integers n and k, let $\mho(n, k)$ be the number of distinct prime divisors of n that are at least k. For example, $\mho(90, 3) = 2$, since the only prime factors of 90 that are at least 3 are 3 and 5. Find the closest integer to

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{\mho(n, k)}{3^{n+k-7}}.$$

Proposed by Daniel Zhu.

---

5 A positive integer N is *piquant* if there exists a positive integer m such that if n_i denotes the number of digits in m^i (in base 10), then $n_1 + n_2 + \cdots + n_{10} = N$. Let p_M denote the fraction of the first M positive integers that are piquant. Find $\lim\limits_{M\to\infty} p_M$.

Proposed by James Lin.

---

6 A polynomial $P(x)$ is a *base-n polynomial* if it is of the form $a_dx^d + a_{d-1}x^{d-1} + \cdots + a_1x + a_0$, where each a_i is an integer between 0 and $n-1$ inclusive and $a_d > 0$. Find the largest positive integer n such that for any real number c, there exists at most one base-n polynomial $P(x)$ for which $P(\sqrt{2} + \sqrt{3}) = c$.

Proposed by James Lin.

---

7 Find the sum of all positive integers n for which

$$\frac{15 \cdot n!^2 + 1}{2n - 3}$$

is an integer.

Proposed by Andrew Gu.

---

8 Let $P(x)$ be the unique polynomial of degree at most 2020 satisfying $P(k^2) = k$ for $k = 0, 1, 2, \ldots, 2020$. Compute $P(2021^2)$.

Proposed by Milan Haiman.

---

9 Let $P(x) = x^{2020} + x + 2$, which has 2020 distinct roots. Let $Q(x)$ be the monic polynomial of degree $\binom{2020}{2}$ whose roots are the pairwise products of the roots of $P(x)$. Let α satisfy $P(\alpha) = 4$. Compute the sum of all possible values of $Q(\alpha^2)^2$.

Proposed by Milan Haiman.

---

10 We define $\mathbb{F}_{101}[x]$ as the set of all polynomials in x with coefficients in $\mathbb{F}_{101}$ (the integers modulo 101 with usual addition and subtraction), so that two polynomials are equal if and only if the coefficients of x^k are equal in $\mathbb{F}_{101}$ for each nonnegative integer k. For example, $(x+3)(100x+5) = 100x^2 + 2x + 15$ in $\mathbb{F}_{101}[x]$ because the corresponding coefficients are equal modulo 101.

We say that $f(x) \in \mathbb{F}_{101}[x]$ is *lucky* if it has degree at most 1000 and there exist $g(x), h(x) \in \mathbb{F}_{101}[x]$ such that

$$f(x) = g(x)(x^{1001} - 1) + h(x)^{101} - h(x)$$

in $\mathbb{F}_{101}[x]$. Find the number of lucky polynomials.

Proposed by Michael Ren.

– Combinatorics

---

1 How many ways can the vertices of a cube be colored red or blue so that the color of each vertex is the color of the majority of the three vertices adjacent to it?

Proposed by Milan Haiman.

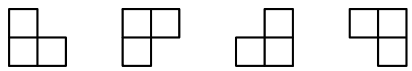
---

2 How many positive integers at most 420 leave different remainders when divided by each of 5, 6, and 7?

Proposed by Milan Haiman.

---

3 Each unit square of a 4×4 square grid is colored either red, green, or blue. Over all possible colorings of the grid, what is the maximum possible number of L-trominos that contain exactly one square of each color? (L-trominos are made up of three unit squares sharing a corner, as shown below.)



Proposed by Andrew Lin.

---

4 Given an 8×8 checkerboard with alternating white and black squares, how many ways are there to choose four black squares and four white squares so that no two of the eight chosen squares are in the same row or column?

Proposed by James Lin.

---

5 Let S be a set of intervals defined recursively as follows:

- Initially, $[1, 1000]$ is the only interval in S.
-If $l \neq r$ and $[l, r] \in S$, then both $\left[l, \left\lfloor \frac{l+r}{2} \right\rfloor\right], \left[\left\lfloor \frac{l+r}{2} \right\rfloor + 1, r\right] \in S$.

(Note that S can contain intervals such as $[1, 1]$, which contain a single integer.) An integer i is chosen uniformly at random from the range $[1, 1000]$. What is the expected number of intervals in S which contain i?

Proposed by Benjamin Qi.

---

6 Alice writes 1001 letters on a blackboard, each one chosen independently and uniformly at random from the set $S = \{a, b, c\}$. A move consists of erasing two distinct letters from the board and replacing them with the third letter in S. What is the probability that Alice can perform a sequence of moves which results in one letter remaining on the blackboard?

Proposed by Daniel Zhu.

---

7 Anne-Marie has a deck of 16 cards, each with a distinct positive factor of 2002 written on it. She shuffles the deck and begins to draw cards from the deck without replacement. She stops when there exists a nonempty subset of the cards in her hand whose numbers multiply to a perfect square. What is the expected number of cards in her hand when she stops?

Proposed by Michael Ren.

---

8 Let Γ_1 and Γ_2 be concentric circles with radii 1 and 2, respectively. Four points are chosen on the circumference of Γ_2 independently and uniformly at random, and are then connected to form a convex quadrilateral. What is the probability that the perimeter of this quadrilateral intersects Γ_1?

Proposed by Yuan Yao.

---

9 Farmer James wishes to cover a circle with circumference 10π with six different types of colored arcs. Each type of arc has radius 5, has length either π or 2π, and is colored either red, green, or blue. He has an unlimited number of each of the six arc types. He wishes to completely cover his circle without overlap, subject to the following conditions:

- Any two adjacent arcs are of different colors.
- Any three adjacent arcs where the middle arc has length π are of three different colors.

Find the number of distinct ways Farmer James can cover his circle. Here, two coverings are equivalent if and only if they are rotations of one another. In particular, two colorings are considered distinct if they are reflections of one another, but not rotations of one another.

Proposed by James Lin.

---

10 Max repeatedly throws a fair coin in a hurricane. For each throw, there is a 4% chance that the coin gets blown away. He records the number of heads H and the number of tails T before the coin is lost. (If the coin is blown away on a toss, no result is recorded for that toss.) What is the expected value of $|H-T|$?

Proposed by Krit Boonsiriseth.

– Geometry

---

1 Let $DIAL$, FOR, and $FRIEND$ be regular polygons in the plane. If $ID=1$, find the product of all possible areas of OLA.

Proposed by Andrew Gu.

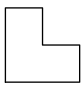
---

2 Let ABC be a triangle with $AB=5$, $AC=8$, and $\angle BAC=60^\circ$. Let $UVWXYZ$ be a regular hexagon that is inscribed inside ABC such that U and V lie on side BA, W and X lie on side AC, and Z lies on side CB. What is the side length of hexagon $UVWXYZ$?

Proposed by Ryan Kim.

---

3 Consider the L-shaped tromino below with 3 attached unit squares. It is cut into exactly two pieces of equal area by a line segment whose endpoints lie on the perimeter of the tromino. What is the longest possible length of the line segment?



Proposed by James Lin.

---

4 Let $ABCD$ be a rectangle and E be a point on segment AD. We are given that quadrilateral $BCDE$ has an inscribed circle ω_1 that is tangent to BE at T. If the incircle ω_2 of ABE is also tangent to BE at T, then find the ratio of the radius of ω_1 to the radius of ω_2.

Proposed by James Lin.

---

5 Let $ABCDEF$ be a regular hexagon with side length 2. A circle with radius 3 and center at A is drawn. Find the area inside quadrilateral $BCDE$ but outside the circle.

Proposed by Carl Joshua Quines.

---

6 Let ABC be a triangle with $AB = 5$, $BC = 6$, $CA = 7$. Let D be a point on ray AB beyond B such that $BD = 7$, E be a point on ray BC beyond C such that $CE = 5$, and F be a point on ray CA beyond A such that $AF = 6$. Compute the area of the circumcircle of DEF.

Proposed by James Lin.

---

7 Let Γ be a circle, and ω_1 and ω_2 be two non-intersecting circles inside Γ that are internally tangent to Γ at X_1 and X_2, respectively. Let one of the common internal tangents of ω_1 and ω_2 touch ω_1 and ω_2 at T_1 and T_2, respectively, while intersecting Γ at two points A and B. Given that $2X_1T_1 = X_2T_2$ and that ω_1, ω_2, and Γ have radii 2, 3, and 12, respectively, compute the length of AB.

Proposed by James Lin.

---

8 Let ABC be an acute triangle with circumcircle Γ. Let the internal angle bisector of $\angle BAC$ intersect BC and Γ at E and N, respectively. Let A' be the antipode of A on Γ and let V be the point where AA' intersects BC. Given that $EV = 6$, $VA' = 7$, and $A'N = 9$, compute the radius of Γ.

Proposed by James Lin.

---

9 Circles $\omega_a, \omega_b, \omega_c$ have centers A, B, C, respectively and are pairwise externally tangent at points D, E, F (with $D \in BC, E \in CA, F \in AB$). Lines BE and CF meet at T. Given that ω_a has radius 341, there exists a line ℓ tangent to all three circles, and there exists a circle of radius 49 tangent to all three circles, compute the distance from T to ℓ.

Proposed by Andrew Gu.

---

10 Let Γ be a circle of radius 1 centered at O. A circle Ω is said to be *friendly* if there exist distinct circles ω_1, ω_2, ..., ω_{2020}, such that for all $1 \leq i \leq 2020$, ω_i is tangent to Γ, Ω, and ω_{i+1}. (Here, $\omega_{2021} = \omega_1$.) For each point P in the plane, let $f(P)$ denote the sum of the areas of all friendly circles centered at P. If A and B are points such that $OA = \frac{1}{2}$ and $OB = \frac{1}{3}$, determine $f(A) - f(B)$.

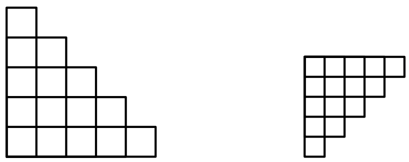
Proposed by Michael Ren.

– Team

1 Let n be a positive integer. Define a sequence by $a_0 = 1$, $a_{2i+1} = a_i$, and $a_{2i+2} = a_i + a_{i+1}$ for each $i \geq 0$. Determine, with proof, the value of $a_0 + a_1 + a_2 + \cdots + a_{2^n-1}$.

Proposed by Kevin Ren.

2 Let n be a fixed positive integer. An n-staircase is a polyomino with $\frac{n(n+1)}{2}$ cells arranged in the shape of a staircase, with arbitrary size. Here are two examples of 5-staircases:



Prove that an n-staircase can be dissected into strictly smaller n-staircases.

Proposed by James Lin.

3 Let ABC be a triangle inscribed in a circle ω and ℓ be the tangent to ω at A. The line through B parallel to AC meets ℓ at P, and the line through C parallel to AB meets ℓ at Q. The circumcircles of ABP and ACQ meet at $S \neq A$. Show that AS bisects BC.

Proposed by Andrew Gu.

4 Alan draws a convex 2020-gon $\mathcal{A} = A_1A_2 \cdots A_{2020}$ with vertices in clockwise order and chooses 2020 angles $\theta_1, \theta_2, \dots, \theta_{2020} \in (0, \pi)$ in radians with sum 1010π. He then constructs isosceles triangles $\triangle A_iB_iA_{i+1}$ on the exterior of $\mathcal{A}$ with $B_iA_i = B_iA_{i+1}$ and $\angle A_iB_iA_{i+1} = \theta_i$. (Here, $A_{2021} = A_1$.) Finally, he erases $\mathcal{A}$ and the point B_1. He then tells Jason the angles $\theta_1, \theta_2, \dots, \theta_{2020}$ he chose. Show that Jason can determine where B_1 was from the remaining 2019 points, i.e. show that B_1 is uniquely determined by the information Jason has.

Proposed by Andrew Gu.

5 Let a_0, b_0, c_0, a, b, c be integers such that $\gcd(a_0, b_0, c_0) = \gcd(a, b, c) = 1$. Prove that there exists a positive integer n and integers $a_1, a_2, \dots, a_n = a, b_1, b_2, \dots, b_n = b, c_1, c_2, \dots, c_n = c$ such that for all $1 \leq i \leq n$, $a_{i-1}a_i + b_{i-1}b_i + c_{i-1}c_i = 1$.

Proposed by Michael Ren.

6 Let $n > 1$ be a positive integer and S be a collection of $\frac{1}{2}\binom{2n}{n}$ distinct n-element subsets of $\{1, 2, \dots, 2n\}$. Show that there exists $A, B \in S$ such that $|A \cap B| \le 1$.

Proposed by Michael Ren.

---

7 Positive real numbers x and y satisfy

$$\left|\left|\cdots\left|\left||x|-y\right|-x\right|\cdots-y\right|-x\right| = \left|\left|\cdots\left|\left||y|-x\right|-y\right|\cdots-x\right|-y\right|$$

where there are 2019 absolute value signs $|\cdot|$ on each side. Determine, with proof, all possible values of $\frac{x}{y}$.

Proposed by Krit Boonsiriseth.

---

8 Let ABC be a scalene triangle with angle bisectors AD, BE, and CF so that D, E, and F lie on segments BC, CA, and AB respectively. Let M and N be the midpoints of BC and EF respectively. Prove that line AN and the line through M parallel to AD intersect on the circumcircle of ABC if and only if $DE = DF$.

Proposed by Michael Ren.

---

9 Let $p > 5$ be a prime number. Show that there exists a prime number $q < p$ and a positive integer n such that p divides $n^2 - q$.

Proposed by Andrew Gu.

---

10 Let n be a fixed positive integer, and choose n positive integers $a_1, \dots, a_n$. Given a permutation π on the first n positive integers, let $S_\pi = \{i \mid \frac{a_i}{\pi(i)} \text{ is an integer}\}$. Let N denote the number of distinct sets S_π as π ranges over all such permutations. Determine, in terms of n, the maximum value of N over all possible values of $a_1, \dots, a_n$.

Proposed by James Lin.
