

Israel Joseph Gillis Mathematical Olympiad 1997www.artofproblemsolving.com/community/c1075217

by parmenides51

- 1 Find all real solutions to the system of equations $x^2 + y^2 = 6z y^2 + z^2 = 6x z^2 + x^2 = 6y$.

- 2 We are given a balance with two bowls and a number of weights.
 - (a) Give an example of four integer weights using which one can measure any weight of $1, 2, \dots, 40$ grams.
 - (b) Are there four weights using which one can measure any weight of $1, 2, \dots, 50$ grams?

- 3 Let $n?$ denote the product of all primes smaller than n .
Prove that $n? > n$ holds for any natural number $n > 3$.

- 4 Let $f : [0, 1] \rightarrow [0, 1]$ be a continuous, strictly increasing function such that $f(0) = 0$ and $f(1) = 1$. Prove that
$$f\left(\frac{1}{10}\right) + f\left(\frac{2}{10}\right) + \dots + f\left(\frac{9}{10}\right) + f^{-1}\left(\frac{1}{10}\right) + \dots + f^{-1}\left(\frac{9}{10}\right) \leq \frac{99}{10}$$

- 5 The natural numbers $a_1, a_2, \dots, a_n, n \geq 12$, are smaller than $9n^2$ and pairwise coprime.
Show that at least one of these numbers is prime.

- 6 In a certain country, every two cities are connected either by an airline route or by a railroad.
Prove that one can always choose a type of transportation in such a way that each city can be reached from any other city with at most two transfers.

- 7 A square with side 10^6 , with a corner square with side 10^{-3} cut off, is partitioned into 10 rectangles. Prove that at least one of these rectangles has the ratio of the greater side to the smaller one at least 9.

- 8 Two equal circles are internally tangent to a larger circle at A and B . Let M be a point on the larger circle, and let lines MA and MB intersect the corresponding smaller circles at A' and B' . Prove that $A'B'$ is parallel to AB .