Romania Team Selection Test 1989

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– BMO TST

1 Let M denote the set of $m \times n$ matrices with entries in the set $\{0, 1, 2, 3, 4\}$ such that in each row and each column the sum of elements is divisible by 5. Find the cardinality of set M.

2 Let P be a point on a circle C and let ϕ be a given angle incommensurable with 2π. For each $n \in N, P_n$ denotes the image of P under the rotation about the center O of C by the angle $\alpha_n = n\phi$. Prove that the set $M = \{P_n | n \ge 0\}$ is dense in C.

3 Let $ABCD$ be a parallelogram and M, N be points in the plane such that $C \in (AM)$ and $D \in (BN)$. Lines NA, NC meet lines MB, MD at points E, F, G, H. Show that points E, F, G, H lie on a circle if and only if $ABCD$ is a rhombus.

4 A family of finite sets $\{A_1, A_2, ..., A_m\}$is called *equipartitionable* if there is a function $\varphi : \cup_{i=1}^{m} \to \{-1, 1\}$ such that $\sum_{x \in A_i} \varphi(x) = 0$ for every $i = 1, ..., m$. Let $f(n)$ denote the smallest possible number of n-element sets which form a non-equipartitionable family. Prove that
a) $f(4k + 2) = 3$ for each nonnegative integer k,
b) $f(2n) \le 1 + md(n)$, where $md(n)$ denotes the least positive non-divisor of n.

5 A laticial cycle of length n is a sequence of lattice points (x_k, y_k), $k = 0, 1, \cdots, n$, such that $(x_0, y_0) = (x_n, y_n) = (0, 0)$ and $|x_{k+1} - x_k| + |y_{k+1} - y_k| = 1$ for each k. Prove that for all n, the number of latticial cycles of length n is a perfect square.

– IMO TST

– Day 1

1 Let the sequence (a_n) be defined by $a_n = n^6 + 5n^4 - 12n^2 - 36, n \ge 2$.
(a) Prove that any prime number divides some term in this sequence.
(b) Prove that there is a positive integer not dividing any term in the sequence.
(c) Determine the least $n \ge 2$ for which $1989 | a_n$.

2 Find all monic polynomials $P(x), Q(x)$ with integer coefficients such that $Q(0) = 0$ and $P(Q(x)) = (x - 1)(x - 2)...(x - 15)$.

3 Find all pair (m, n) of integer ($m > 1, n \ge 3$) with the following property:If an n-gon can be partitioned into m isoceles triangles,then the n-gon has two congruent sides.

4 Let r, n be positive integers. For a set A, let $\binom{A}{r}$ denote the family of all r-element subsets of A. Prove that if A is infinite and $f : \binom{A}{r} \to 1, 2, ..., n$ is any function, then there exists an infinite subset B of A such that $f(X) = f(Y)$ for all $X, Y \in \binom{B}{r}$.

– Day 2

1 Let F be the set of all functions $f : N \to N$ which satisfy $f(f(x)) - 2f(x) + x = 0$ for all $x \in N$. Determine the set $A = \{f(1989)|f \in F\}$.

2 Let a, b, c be coprime nonzero integers. Prove that for any coprime integers u, v, w with $au + bv + cw = 0$ there exist integers m, n, p such that $a = nw - pv, b = pu - mw, c = mv - nu$.

3 (a) Find the point M in the plane of triangle ABC for which the sum $MA + MB + MC$ is minimal.
(b) Given a parallelogram $ABCD$ whose angles do not exceed 120^o, determine $min\{MA + MB + NC + ND + MN|M, N$ are in the plane $ABCD\}$ in terms of the sides and angles of the parallelogram.

4 Let A, B, C be variable points on edges OX, OY, OZ of a trihedral angle $OXYZ$, respectively. Let $OA = a, OB = b, OC = c$ and R be the radius of the circumsphere S of $OABC$.
Prove that if points A, B, C vary so that $a + b + c = R + l$, then the sphere S remains tangent to a fixed sphere.

– Day 3

1 Prove that $\sqrt{1 + \sqrt{2 + \ldots + \sqrt{n}}} < 2, \forall n \geq 1$.

2 The sequence (a_n) is defined by $a_1 = a_2 = 1, a_3 = 199$ and $a_{n+1} = \frac{1989+a_n a_{n-1}}{a_{n-2}}$ for all $n \geq 3$. Prove that all terms of the sequence are positive integers

3 Let F be the boundary and M, N be any interior points of a triangle ABC. Consider the function $f_{M,N} : F \to R$ defined by $f_{M,N}(P) = MP^2 + NP^2$ and let $\eta_{M,N}$ be the number of points P for which fM, N attains its minimum.
(a) Prove that $1 \leq \eta_{M,N} \leq 3$.
(b) If M is fixed, find the locus of N for which $\eta_{M,N} > 1$.
(c) Prove that the locus of M for which there are points N such that $\eta_{M,N} = 3$ is the interior of a tangent hexagon.