Switzerland Team Selection Test 1998

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by parmenides51

–	Day 1
1	A function $f : R - \{0\} \to R$ has the following properties: (i) $f(x) - f(y) = f(x)f\left(\frac{1}{y}\right) - f(y)f\left(\frac{1}{x}\right)$ for all $x, y \neq 0$, (ii) f takes the value $\frac{1}{2}$ at least once. Determine $f(-1)$. Prove that f is a periodic function
2	Find all nonnegative integer solutions (x, y, z) of the equation $\frac{1}{x+2} + \frac{1}{y+2} = \frac{1}{2} + \frac{1}{z+2}$
3	Given positive numbers a, b, c, find the minimum of the function $f(x) = \sqrt{a^2 + x^2} + \sqrt{(b - x)^2 + c^2}$.
4	Find all numbers n for which it is possible to cut a square into n smaller squares.
5	Points A and B are chosen on a circle k. Let AP and BQ be segments of the same length tangent to k, drawn on different sides of line AB. Prove that the line AB bisects the segment PQ.
–	Day 2
6	Find all prime numbers p for which $p^2 + 11$ has exactly six positive divisors.
7	Consider an $n \times n$ matrix whose entry at the intersection of the i-th row and the $j-$th column equals $i + j - 1$. What is the largest possible value of the product of n entries of the matrix, no two of which are in the same row or column?
8	Let $\triangle\ ABC$ be an equilateral triangle and let P be a point in its interior. Let the lines AP, BP, CP meet the sides BC, CA, AB in the points X, Y, Z respectively. Prove that $XY \cdot YZ \cdot ZX \geq XB \cdot YC \cdot ZA$.
9	If x and y are positive numbers, prove the inequality $\frac{x}{x^4+y^2} + \frac{y}{x^2+y^4} \leq \frac{1}{xy}$.
10	5. Let $f : R \to R$ be a function that satisfies for all $x \in R$ (i) $\lvert f(x)\rvert \leq 1$, and

(ii) $f\left(x+\frac{13}{42}\right)+f(x)=f\left(x+\frac{1}{6}\right)+f\left(x+\frac{1}{7}\right)$
Prove that f is a periodic function