Switzerland Team Selection Test 2003

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by parmenides51, ThirdTimeLucky

–	Day 1
1	Real numbers x, y, a satisfy the equations $x+y=x^3+y^3=x^5+y^5=a$. Find all possible values of a.
2	In an acute-angled triangle ABC, E and F are the feet of the altitudes from B and C, and G and H are the projections of B and C on EF, respectively. Prove that $HE=FG$.
3	Find the largest real number C_1 and the smallest real number C_2, such that, for all reals a,b,c,d,e, we have $$C_1<\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+d}+\frac{d}{d+e}+\frac{e}{e+a}<C_2$$
4	Find the largest natural number n that divides $a^{25}-a$ for all integers a.
5	There are n pieces on the squares of a 5×9 board, at most one on each square at any time during the game. A move in the game consists of simultaneously moving each piece to a neighboring square by side, under the restriction that a piece having been moved horizontally in the previous move must be moved vertically and vice versa. Find the greatest value of n for which there exists an initial position starting at which the game can be continued until the end of the world.
–	Day 2
6	Let a,b,c be positive real numbers satisfying $a+b+c=2$. Prove the inequality $$\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ca}\geq\frac{27}{13}$$
7	Find all polynomials $Q(x)=ax^2+bx+c$ with integer coefficients for which there exist three different prime numbers p_1,p_2,p_3 such that $\|Q(p_1)\|=\|Q(p_2)\|=\|Q(p_3)\|=11$.
8	Let $A_1A_2A_3$ be a triangle and ω_1 be a circle passing through A_1 and A_2. Suppose that there are circles $\omega_2,...,\omega_7$ such that: (a) ω_k passes through A_k and A_{k+1} for $k=2,3,...,7$, where $A_i=A_{i+3}$, (b) ω_k and ω_{k+1} are externally tangent for $k=1,2,...,6$. Prove that $\omega_1=\omega_7$.

9 Given integers $0 < a_1 < a_2 < ... < a_{101} < 5050$, prove that one can always choose for different numbers a_k, a_l, a_m, a_n such that $5050|a_k + a_l - a_m - a_n$

10 Find all strictly monotonous functions $f : N \to N$ that satisfy $f(f(n)) = 3n$ for all $n \in N$.