Czech And Slovak Mathematical Olympiad, Round III, Category A 1997

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1 Let ABC be a triangle with sides a, b, c and corresponding angles $\alpha, \beta\gamma$. Prove that if $\alpha = 3\beta$ then $(a^2 - b^2)(a - b) = bc^2$. Is the converse true?

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2 Each side and diagonal of a regular n-gon ($n \geq 3$) is colored red or blue. One may choose a vertex and change the color of all segments emanating from that vertex. Prove that, no matter how the edges were colored initially, one can achieve that the number of blue segments at each vertex is even. Show also that the resulting coloring depends only on the initial coloring.

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3 A tetrahedron $ABCD$ is divided into five polyhedra so that each face of the tetrahedron is a face of (exactly) one polyhedron, and that the intersection of any two of the polyhedra is either a common vertex, a common edge, or a common face. What is the smallest possible sum of the numbers of faces of the five polyhedra?

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4 Show that there exists an increasing sequence $a_1, a_2, a_3, ...$ of natural numbers such that, for any integer $k \geq 2$, the sequence $k + a_n$ ($n \in N$) contains only finitely many primes.

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5 For a given integer $n \geq 2$, find the maximum possible value of $V_n = \sin x_1 \cos x_2 + \sin x_2 \cos x_3 + ... + \sin x_n \cos x_1$, where $x_1, x_2, ..., x_n$ are real numbers.

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6 In a parallelogram $ABCD$, triangle ABD is acute-angled and $\angle BAD = \pi/4$. Consider all possible choices of points K, L, M, N on sides AB, BC, CD, DA respectively, such that $KLMN$ is a cyclic quadrilateral whose circumradius equals those of triangles ANK and CLM. Find the locus of the intersection of the diagonals of $KLMN$
