Czech And Slovak Mathematical Olympiad, Round III, Category A 1996

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1 A sequence $(G_n)_{n=0}^{\infty}$ satisfies $G(0) = 0$ and $G(n) = n - G(G(n-1))$ for each $n \in N$. Show that
(a) $G(k) \geq G(k-1)$ for every $k \in N$;
(b) there is no integer k for which $G(k-1) = G(k) = G(k+1)$.

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2 Let AP, BQ and CR be altitudes of an acute-angled triangle ABC. Show that for any point X inside the triangle PQR there exists a tetrahedron $ABCD$ such that X is the point on the face ABC at the greatest distance from D (measured along the surface of the tetrahedron).

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3 Given six three-element subsets of a finite set X, show that it is possible to color the elements of X in two colors so that none of the given subsets is in one color

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4 Points A and B on the rays CX and CY respectively of an acute angle XCY are given so that $CX < CA = CB < CY$. Construct a line meeting the ray CX and the segments AB, BC at K, L, M, respectively, such that $KA \cdot YB = XA \cdot MB = LA \cdot LB \neq 0$.

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5 For which integers k does there exist a function $f : N \to Z$ such that $f(1995) = 1996$ and $f(xy) = f(x) + f(y) + kf(gcd(x, y))$ for all $x, y \in N$?

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6 Let K, L, M be points on sides AB, BC, CA, respectively, of a triangle ABC such that $AK/AB = BL/BC = CM/CA = 1/3$. Show that if the circumcircles of the triangles AKM, BLK, CML are equal, then so are the incircles of these triangles.
