Czech And Slovak Mathematical Olympiad, Round III, Category A 1994

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1 Let $f : N \to N$ be a function which satisfies $f(x) + f(x + 2) \le 2f(x + 1)$ for any $x \in N$. Prove that there exists a line in the coordinate plane containing infinitely many points of the form $(n, f(n)), n \in N$.

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2 A cuboid of volume V contains a convex polyhedron M. The orthogonal projection of M onto each face of the cuboid covers the entire face. What is the smallest possible volume of polyhedron M?

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3 A convex 1994-gon M is given in the plane. A closed polygonal line consists of 997 of its diagonals. Every vertex is adjacent to exactly one diagonal. Each diagonal divides M into two sides, and the smaller of the numbers of edges on the two sides of M is defined to be the length of the diagonal. Is it posible to have
(a) 991 diagonals of length 3 and 6 of length 2?
(b) 985 diagonals of length $6, 4$ of length 8, and 8 of length 3?

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4 Let $a_1, a_2, ...$ be a sequence of natural numbers such that for each n, the product $(a_n - 1)(a_n - 2)...(a_n - n^2)$ is a positive integral multiple of n^{n^2-1}. Prove that for any finite set P of prime numbers the following inequality holds:

$$\sum_{p \in P} \frac{1}{\log_p a_p} < 1$$

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5 In an acute-angled triangle ABC, the altitudes AA_1, BB_1, CC_1 intersect at point V. If the triangles AC_1V, BA_1V, CB_1V have the same area, does it follow that the triangle ABC is equilateral?

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6 Show that from any four distinct numbers lying in the interval $(0, 1)$ one can choose two distinct numbers a and b such that

$$\sqrt{(1 - a^2)(1 - b^2)} > \frac{a}{2b} + \frac{b}{2a} - ab - \frac{1}{8ab}$$
