Niels Henrik Abels Math Contest (Norwegian Math Olympiad) Final Round 2006

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by parmenides51

1 Each square in an $n \times n$ table is painted black or white. The routes where two rows meet two columns, called a quartet if the remaining squares are the same color.
(a) What is the largest possible number of black squares in a 4×4 table without quartets?
(b) Is it possible to paint a 5×5 table so that it has no quartets?

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2 a) Let a and b be two non-negative real numbers. Show that $a+b \ge \sqrt{\frac{a^2+b^2}{2}} + \sqrt{ab}$

b) Let a and b be two real numbers in $[0,3]$. Show that $\sqrt{\frac{a^2+b^2}{2}} + \sqrt{ab} \ge \frac{(a+b)^2}{2}$

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3 (a) Let a and b be rational numbers such that line $y = ax + b$ intersects the circle $x^2 + y^2 = 5$ at two different points. Show that if one of the intersections has two rational coordinates, so does the other intersection.
(b) Show that there are infinitely many triples (k, n, m) that are such that $k^2 + n^2 = 5m^2$, where k, n and m are integers, and not all three have any in common prime factor.

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4 Let γ be the circumscribed circle about a right-angled triangle ABC with right angle C. Let δ be the circle tangent to the sides AC and BC and tangent to the circle γ internally.
(a) Find the radius i of δ in terms of a when AC and BC both have length a.
(b) Show that the radius i is twice the radius of the inscribed circle of ABC.
