Niels Henrik Abels Math Contest (Norwegian Math Olympiad) Final Round 2005

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by parmenides51

1a A positive integer m is called triangular if $m = 1 + 2 + ... + n$, for an integer n. Show that a positive integer m is triangular if and only if $8m + 1$ is the square of an integer.

1b In a pyramid, the base is a right-angled triangle with integer sides. The height of the pyramid is also integer. Show that the volume of the pyramid is even.

2a In an aquarium there are nine small fish. The aquarium is cube shaped with a side length of two meters and is completely filled with water. Show that it is always possible to find two small fish with a distance of less than $\sqrt{3}$ meters.

2b Let A be the number of all points with integer coordinates in a three-dimensional coordinate system. We assume that nine arbitrary points in A will be colored blue. Show that we can always find two blue dots so that the line segment between them contains at least one point from A.

3a In the isosceles triangle $\triangle ABC$ is $AB = AC$. Let D be the midpoint of the segment BC. The points P and Q are respectively on the lines AD and AB (with $Q \neq B$) so that $PQ = PC$. Show that $\angle PQC = \frac{1}{2}\angle A$

3b In the parallelogram $ABCD$, all sides are equal, and $\angle A = 60^o$. Let F be a point on line AD, H a point on line DC, and G a point on diagonal AC such that $DFGH$ is a parallelogram. Show that then $\triangle BHF$ is equilateral.

4a Show that for all positive real numbers a, b and c, the inequality $(a+b)(a+c) \geq 2\sqrt{abc(a + b + c)}$ is true.

4b Let a, b and c be real numbers such that $ab+bc+ca > a+b+c > 0$. Show then that $a+b+c > 3$
