Niels Henrik Abels Math Contest (Norwegian Math Olympiad) Final Round 2002

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by parmenides51

1a Find all integers k such that both $k+1$ and $16k+1$ are perfect squares.

1b Find all integers c such that the equation $(2a+b)(2b+a)=5^c$ has integer solutions.

2ab a) Let x be a positive real number. Show that $x+1/x \geq 2$.
b) Let $n \geq 2$ be a positive integer and let $x_1, y_1, x_2, y_2, ..., x_n, y_n$ be positive real numbers such that $x_1+x_2+...+x_n \geq x_1y_1+x_2y_2+...+x_ny_n$.

Show that $x_1+x_2+...+x_n \leq \frac{x_1}{y_1}+\frac{x_2}{y_2}+...+\frac{x_n}{y_n}$

2c If a and b are real numbers such that $a^3-3ab^2=8$ and $b^3-3a^2b=11$, then what is a^2+b^2?

3a A circle with center in O is given. Two parallel tangents tangent to the circle at points M and N. Another tangent intersects the first two tangents at points K and L. Show that the circle having the line segment KL as diameter passes through O.

3b Six line segments of lengths $17, 18, 19, 20, 21$ and 23 form the side edges of a triangular pyramid (also called a tetrahedron). Can there exist a sphere tangent to all six lines?

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4 An integer is given $N>1$. Arne and Britt play the following game:
(1) Arne says a positive integer A.
(2) Britt says an integer $B>1$ that is either a divisor of A or a multiple of A. (A itself is a possibility.)
(3) Arne says a new number A that is either $B-1, B$ or $B+1$.
The game continues by repeating steps 2 and 3. Britt wins if she is okay with being told the number N before the 50th has been said. Otherwise, Arne wins.
a) Show that Arne has a winning strategy if $N=10$.
b) Show that Britt has a winning strategy if $N=24$.
c) For which N does Britt have a winning strategy?
