2019 China Northern Math Olympiad

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by minecraftfaq

–	Shijiazhuang, Hebei Province, China
–	Day 1 August 6th, 2019
1	Find all positive intengers x, y, satisfying: $$3^x + x^4 = y! + 2019.$$
2	Two circles O_1 and O_2 intersect at A, B. Diameter AC of $\odot O_1$ intersects $\odot O_2$ at E, Diameter AD of $\odot O_2$ intersects $\odot O_1$ at F. CF intersects O_2 at H, DE intersects O_1 at G, H. $GH \cap O_1 = P$. Prove that $PH = PK$.
3	$n(n \geq 2)$ is a given intenger, and $a_1, a_2, ..., a_n$ are real numbers. For any $i = 1, 2, \cdots, n$, $$a_i \neq -1, a_{i+2} = \frac{a_i^2 + a_i}{a_{i+1} + 1}.$$ Prove: $a_1 = a_2 = \cdots = a_n$. (Note: $a_{n+1} = a_1, a_{n+2} = a_2$.)
4	A manager of a company has 8 workers. One day, he holds a few meetings. (1)Each meeting lasts 1 hour, no break between two meetings. (2)Three workers attend each meeting. (3)Any two workers have attended at least one common meeting. (4)Any worker cannot leave until he finishes all his meetings. Then, how long does the worker who works the longest work at least?
–	Day 2 August 7th, 2019
5	Two circles O_1 and O_2 intersect at A, B. Bisector of outer angle $\angle O_1AO_2$ intersects O_1 at C, O_2 at D. P is a point on $\odot(BCD)$, $CP \cap O_1 = E, DP \cap O_2 = F$. Prove that $PE = PF$.
6	For nonnegative real numbers a, b, c, x, y, z, if$a + b + c = x + y + z = 1$, find the maximum value of $(a - x^2)(b - y^2)(c - z^2)$.
7	There are n cities in Qingqiu Country. The distance between any two cities are different. The king of the country plans to number the cities and set up two-way air lines in such ways:

The first time, set up a two-way air line between city 1 and the city nearest to it.
The second time, set up a two-way air line between city 2 and the city second nearest to it.
...
The $n-1$th time, set up a two-way air line between city $n-1$ and the city farthest to it.
Prove: The king can number the cities in a proper way so that he can go to any other city from any city by plane.

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8 For positive intenger n, define $f(n)$: the smallest positive intenger that does not divide n.
Consider sequence $(a_n): a_1 = a_2 = 1, a_n = a_{f(n)} + 1 (n \geq 3)$.
For example, $a_3 = a_2 + 1 = 2, a_4 = a_3 + 1 = 3$.
(a) Prove that there exists a positive intenger C, for any positive intenger n, $a_n \leq C$.
(b) Are there positive intengers M and T, satisfying that for any positive intenger $n \geq M$, $a_n = a_{n+T}$.
