German National Olympiad 1996, Final Round

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–	Day 1
1	Find all natural numbers n with the following property: Given the decimal writing of n, adding a few digits one can obtain the decimal writing of $1996n$.
2	Let a and b be positive real numbers smaller than 1. Prove that the following two statements are equivalent: (i) $a+b=1$, (ii) Whenever x,y are positive real numbers such that $x<1, y<1, ax+by<1$, the following inequlity holds: $\frac{1}{1-ax-by} \le \frac{a}{1-x}+\frac{b}{1-y}$
3	Let be given an arbitrary tetrahedron $ABCD$ with volume V. Consider all lines which pass through the barycenter S of the tetrahedron and intersect the edges AD, BD, CD at points A', B', C respectively. It is known that among the obtained tetrahedra there exists one with the minimal volume. Express this minimal volume in terms of V
–	Day 2
4	Find all pairs of real numbers (x,y) which satisfy the system $$\begin{cases} x-y=7 \\ \sqrt[3]{x^2}+\sqrt[3]{xy}+\sqrt[3]{y^2}=7 \end{cases}$$
5	Given two non-intersecting chords AB and CD of a circle k and a length $a<CD$. Determine a point X on k with the following property: If lines XA and XB intersect CD at points P and Q respectively, then $PQ=a$. Show how to construct all such points X and prove that the obtained points indeed have the desired property.
6a	Prove the following statement: If a polynomial $p(x)=x^3+Ax^2+Bx+C$ has three real roots at least two of which are distinct, then $A^2+B^2+18C>0$.
6b	Each point of a plane is colored in one of three colors: red, black and blue. Prove that there exists a rectangle in this plane whose vertices all have the same color.

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