German National Olympiad 1997, Final Round

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–	Day 1
1	Prove that there are no perfect squares a, b, c such that $ab - bc = a$.
2	For a positive integer k, let us denote by $u(k)$ the greatest odd divisor of k. Prove that, for each $n \in N$, $\frac{1}{2^n}\sum_{k=1}^{2^n}\frac{u(k)}{k} > \frac{2}{3}$.
3	In a convex quadrilateral $ABCD$ we are given that $\angle CBD = 10^o$, $\angle CAD = 20^o$, $\angle ABD = 40^o$, $\angle BAC = 50^o$. Determine the angles $\angle BCD$ and $\angle ADC$.
–	Day 2
4	Find all real solutions (x, y, z) of the system of equations $$\begin{cases} x^3 = 2y - 1 \\ y^3 = 2z - 1 \\ z^3 = 2x - 1 \end{cases}$$
5	We are given n discs in a plane, possibly overlapping, whose union has the area 1. Prove that we can choose some of them which are mutually disjoint and have the total area greater than $1/9$.
6a	Let us define f and g by $f(x) = x^5 + 5x^4 + 5x^3 + 5x^2 + 1$, $g(x) = x^5 + 5x^4 + 3x^3 - 5x^2 - 1$. Determine all prime numbers p such that, for at least one integer $x, 0 \le x < p-1$, both $f(x)$ and $g(x)$ are divisible by p. For each such p, find all x with this property.
6b	An approximate construction of a regular pentagon goes as follows. Inscribe an arbitrary convex pentagon $P_1P_2P_3P_4P_5$ in a circle. Now choose an arror bound $\epsilon > 0$ and apply the following procedure. (a) Denote $P_0 = P_5$ and $P_6 = P_1$ and construct the midpoint Q_i of the circular arc $P_{i-1}P_{i+1}$ containing P_i. (b) Rename the vertices $Q_1, ..., Q_5$ as $P_1, ..., P_5$. (c) Repeat this procedure until the difference between the lengths of the longest and the shortest among the arcs P_iP_{i+1} is less than ϵ. Prove this procedure must end in a finite time for any choice of ϵ and the points P_i.