German National Olympiad 1998, Final Round

www.artofproblemsolving.com/community/c1080068

by parmenides51, Kezer

–	Day 1
1	Find all possible numbers of lines in a plane which intersect in exactly 37 points.
2	Two pupils A and B play the following game. They begin with a pile of 1998 matches and A plays first. A player who is on turn must take a nonzero square number of matches from the pile. The winner is the one who makes the last move. Decide who has the winning strategy and give one such strategy.
3	For each nonnegative integer k find all nonnegative integers x, y, z such that $x^2 + y^2 + z^2 = 8^k$
–	Day 2
4	Let a be a positive real number. Then prove that the polynomial $$p(x) = a^3x^3 + a^2x^2 + ax + a$$ has integer roots if and only if $a = 1$ and determine those roots.
5	A sequence (a_n) is given by $a_0 = 0, a_1 = 1$ and $a_{k+2} = a_{k+1} + a_k$ for all integers $k \geq 0$. Prove that the inequality $\sum_{k=0}^{n} \frac{a_k}{2^k} < 2$ holds for all positive integers n.
6a	Find all real pairs (x, y) that solve the system of equations $$x^5 = 21x^3 + y^3 \quad (1)$$ $$y^5 = x^3 + 21y^3. \quad (2)$$
6b	Prove that the following statement holds for all odd integers $n \geq 3$: If a quadrilateral $ABCD$ can be partitioned by lines into n cyclic quadrilaterals, then $ABCD$ is itself cyclic.