German National Olympiad 1999, Final Round

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–	Day 1
1	Find all x, y which satisfy the equality $x^2 + xy + y^2 = 97$, when x, y are a) natural numbers, b) integers .
2	Determine all real numbers x for which $1 + \frac{x}{2} - \frac{x^2}{8} \le \sqrt{1+x} \le 1 + \frac{x}{2}$
3	A mathematician investigates methods of finding area of a convex quadrilateral obtains the following formula for the area A of a quadrilateral with consecutive sides a, b, c, d: $A = \frac{a+c}{2}\frac{b+d}{2}$ (1) and $A = \sqrt{(p-a)(p-b)(p-c)(p-d)}$ (2) where $p = (a+b+c+d)/2$. However, these formulas are not valid for all convex quadrilaterals. Prove that (1) holds if and only if the quadrilateral is a rectangle, while (2) holds if and only if the quadrilateral is cyclic.
–	Day 2
4	A convex polygon P is placed inside a unit square Q. Prove that the perimeter of P does not exceed 4.
5	Consider the following inequality for real numbers x, y, z: $\lvert x-y\rvert+\lvert y-z\rvert+\lvert z-x\rvert \le a\sqrt{x^2+y^2+z^2}$. (a) Prove that the inequality is valid for $a = 2\sqrt{2}$ (b) Assuming that x, y, z are nonnegative, show that the inequality is also valid for $a = 2$.
6a	Suppose that an isosceles right-angled triangle is divided into m acute-angled triangles. Find the smallest possible m for which this is possible.
6b	Determine all pairs (m, n) of natural numbers for which $4^m + 5^n$ is a perfect square.