Iran Team Selection Test 2020

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by Tintarn, matinyousefi, Dadgarnia

Test 1 Day 1

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1 A weighted complete graph with distinct positive wights is given such that in every triangle is *degenerate* that is wight of an edge is equal to sum of two other. Prove that one can assign values to the vertexes of this graph such that the wight of each edge is the difference between two assigned values of the endpoints.

Proposed by Morteza Saghafian

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2 Let O be the circumcenter of the triangle ABC. Points D, E are on sides AC, AB and points P, Q, R, S are given in plane such that P, C and R, C are on different sides of AB and pints Q, B and S, B are on different sides of AC such that R, S lie on circumcircle of DAP, EAQ and $\triangle BCE \sim \triangle ADQ, \triangle CBD \sim \triangle AEP$(In that order), $\angle ARE = \angle ASD = \angle BAC$, If $RS \| PQ$ prove that RE, DS are concurrent on AO.

Proposed by Alireza Dadgarnia

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3 We call a number n *interesting* if for each permutation σ of $1, 2, \ldots, n$ there exist polynomials $P_1, P_2, \ldots, P_n$ and $\epsilon > 0$ such that: $i)$ $P_1(0) = P_2(0) = \ldots = P_n(0)$ $ii)$ $P_1(x) > P_2(x) > \ldots > P_n(x)$ for $-\epsilon < x < 0$ $iii)$ $P_{\sigma(1)}(x) > P_{\sigma(2)}(x) > \ldots > P_{\sigma(n)}(x)$ for $0 < x < \epsilon$
Find all *interesting* n.

Proposed by Mojtaba Zare Bidaki

Test 1 Day 2

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4 Given a function $g : [0, 1] \to \mathbb{R}$ satisfying the property that for every non empty dissection of the trivial $[0, 1]$ to subsets A, B we have either $\exists x \in A; g(x) \in B$ or $\exists x \in B; g(x) \in A$ and we have furthermore $g(x) > x$ for $x \in [0, 1]$. Prove that there exist infinite $x \in [0, 1]$ with $g(x) = 1$.

Proposed by Ali Zamani

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5 Given $k \in \mathbb{Z}$ prove that there exist infinite pairs of distinct natural numbers such that

$$n + s(2n) = m + s(2m)$$
$$kn + s(n^2) = km + s(m^2).$$

($s(n)$ denotes the sum of digits of n.)

Proposed by Mohammadamin Sharifi

6 n positive numbers are given. Is it always possible to find a convex polygon with $n+3$ edges and a triangulation of it so that the length of the diameters used in the triangulation are the given n numbers?

Proposed by Morteza Saghafian

Test 2 Day 1

1 We call a monic polynomial $P(x) \in \mathbb{Z}[x]$ *square-free mod n* if there dose not exist polynomials $Q(x), R(x) \in \mathbb{Z}[x]$ with Q being non-constant and $P(x) \equiv Q(x)^2 R(x) \mod n$. Given a prime p and integer $m \geq 2$. Find the number of monic *square-free mod p* $P(x)$ with degree m and coeeficients in $\{0, 1, 2, 3, ..., p-1\}$.

Proposed by Masud Shafaie

2 Alice and Bob take turns alternatively on a 2020×2020 board with Alice starting the game. In every move each person colours a cell that have not been coloured yet and will be rewarded with as many points as the coloured cells in the same row and column. When the table is coloured completely, the points determine the winner. Who has the wining strategy and what is the maximum difference he/she can grantees?

Proposed by Seyed Reza Hosseini

3 Given a triangle ABC with circumcircle Γ. Points E and F are the foot of angle bisectors of B and C, I is incenter and K is the intersection of AI and EF. Suppose that T be the midpoint of arc BAC. Circle Γ intersects the A-median and circumcircle of AEF for the second time at X and S. Let S' be the reflection of S across AI and J be the second intersection of circumcircle of $AS'K$ and AX. Prove that quadrilateral $TJIX$ is cyclic.

Proposed by Alireza Dadgarnia and Amir Parsa Hosseini

Test 2 Day 2

4 Let ABC be an isosceles triangle ($AB = AC$) with incenter I. Circle ω passes through C and I and is tangent to AI. ω intersects AC and circumcircle of ABC at Q and D, respectively. Let M be the midpoint of AB and N be the midpoint of CQ. Prove that AD, MN and BC are concurrent.

Proposed by Alireza Dadgarnia

5 For every positive integer $k > 1$ prove that there exist a real number x so that for every positive integer $n < 1398$:

$$\{x^n\} < \{x^{n-1}\} \Longleftrightarrow k \mid n.$$

Proposed by Mohammad Amin Sharifi

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6 p is an odd prime number. Find all $\frac{p-1}{2}$-tuples $\left(x_1, x_2, \dots, x_{\frac{p-1}{2}}\right) \in \mathbb{Z}_p^{\frac{p-1}{2}}$ such that

$$\sum_{i=1}^{\frac{p-1}{2}} x_i \equiv \sum_{i=1}^{\frac{p-1}{2}} x_i^2 \equiv \cdots \equiv \sum_{i=1}^{\frac{p-1}{2}} x_i^{\frac{p-1}{2}} \pmod{p}.$$

Proposed by Ali Partofard
