2016 China Northern Math Olympiad

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by minecraftfaq

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–	Grade 10
1	$a_1, a_2, \cdots, a_n$ are positive real numbers, $a_1 + a_2 + \cdots, a_n = 1$. Prove that $$\sum_{m=1}^{n} \frac{a_m}{\prod\limits_{k=1}^{m} (1+a_k)} \leq 1 - \frac{1}{2^n}.$$
2	In isosceles triangle ABC, $\angle CAB = \angle CBA = \alpha$, points P, Q are on different sides of line AB, and $\angle CAP = \angle ABQ = \beta$, $\angle CBP = \angle BAQ = \gamma$. Prove that P, C, Q are colinear.
3	Prove: **(a)** There are infinitely many positive intengers n, satisfying: $$\gcd(n, [\sqrt{2}n]) = 1.$$ **(b)** There are infinitely many positive intengers n, satisfying: $$\gcd(n, [\sqrt{2}n]) > 1.$$
4	Can we put intengers $1, 2, \cdots, 12$ on a circle, number them $a_1, a_2, \cdots, a_{12}$ in order. For any $1 \leq i < j \leq 12$, $\lvert a_i - a_j \rvert \neq \lvert i - j \rvert$?
5	Let $\theta_i \in (0, \frac{\pi}{2}) (i = 1, 2, \cdots, n)$. Prove: $$(\sum_{i=1}^{n} \tan\theta_i)(\sum_{i=1}^{n} \cot\theta_i) \geq (\sum_{i=1}^{n} \sin\theta_i)^2 + (\sum_{i=1}^{n} \cos\theta_i)^2.$$
6	Four points B, E, A, F lie on line AB in order, four points C, G, D, H lie on line CD in order, satisfying: $$\frac{AE}{EB} = \frac{AF}{FB} = \frac{DG}{GC} = \frac{DH}{HC} = \frac{AD}{BC}.$$ Prove that $FH \perp EG$.

7 Define sequence $(a_n) : a_n = 2^n + 3^n + 6^n + 1 (n \in \mathbb{Z}_+)$.
Are there intenger $k \geq 2$, satisfying that $\gcd(k, a_i) = 1$ for all $k \in \mathbb{Z}_+$?
If yes, find the smallest k. If not, prove this.

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8 Set $A = \{1, 2, \cdots, n\}$. If there exists nonempty sets B, C, such that $B \cap C = \varnothing, B \cup C = A$. Sum of Squares of all elements in B is M, Sum of Squares of all elements in C is N, $M - N = 2016$. Find the minimum value of n.

– Grade 11

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1 The same as Grade 10, Problem 1.

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2 Inscribed Triangle ABC on circle $\odot O$. Bisector of $\angle ABC$ intersects $\odot O$ at D. Two lines PB and PC that are tangent to $\odot O$ intersect at P. PD intersects AC at E, $\odot O$ at F. M is the midpoint of BC. Prove that M, F, C, E are concyclic.

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3 $m(m > 1)$ is an intenger, define (a_n): $a_0 = m, a_n = \varphi(a_{n-1})$ for all positive intenger n.
If for all nonnegative intenger k, $a_{k+1} \mid a_k$, find all m that is not larger than 2016.
Note: $\varphi(n)$ means Euler Function.

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4 The same as Grade 10, Problem 4.

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5 $a_1 = 2, a_{n+1} = \frac{2^{n+1} a_n}{(n + \frac{1}{2}) a_n + 2^n} (n \in \mathbb{Z}_+)$
(a) Find a_n.
(b) Let $b_n = \frac{n^3 + 2n^2 + 2n + 2}{n(n+1)(n^2+1) a_n}$.
Find $S_n = \sum_{i=1}^{n} b_i$.

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6 The same as Grade 10, Problem 6.

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7 The same as Grade 10, Problem 7.

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8 Given a set $I = \{(x_1, x_2, x_3, x_4) | x_i \in \{1, 2, \cdots, 11\}\}$. $A \subseteq I$, satisfying that for any $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3,$
A, there exists $i, j (1 \leq i < j \leq 4)$, $(x_i - x_j)(y_i - y_j) < 0$. Find the maximum value of $|A|$.
