Finals 1983

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–	Day 1
1	On the plane are given a convex n-gon $P_1P_2....P_n$ and a point Q inside it, not lying on any of its diagonals. Prove that if n is even, then the number of triangles $P_iP_jP_k$ containing the point Q is even.
2	Let be given an irrational number a in the interval $(0,1)$ and a positive integer N. Prove that there exist positive integers p,q,r,s such that $\frac{p}{q} < a < \frac{r}{s}$, $\frac{r}{s} - \frac{p}{q} < \frac{1}{N}$, and $rq - ps = 1$.
3	Consider the following one-player game on an infinite chessboard. If two horizontally or vertically adjacent squares are occupied by a pawn each, and a square on the same line that is adjacent to one of them is empty, then it is allowed to remove the two pawns and place a pawn on the third (empty) square. Prove that if in the initial position all the pawns were forming a rectangle with the number of squares divisible by 3, then it is not possible to end the game with only one pawn left on the board.
–	Day 2
4	Prove that if natural numbers a,b,c,d satisfy the equality $ab = cd$, then $\frac{gcd(a,c)gcd(a,d)}{gcd(a,b,c,d)} = a$
5	On the plane are given unit vectors $\overrightarrow{a_1}, \overrightarrow{a_2}, \overrightarrow{a_3}$. Show that one can choose numbers $c_1, c_2, c_3 \in \{-1, 1\}$ such that the length of the vector $c_1\overrightarrow{a_1} + c_2\overrightarrow{a_2} + c_3\overrightarrow{a_3}$ is at least 2.
6	Prove that if all dihedral angles of a tetrahedron are acute, then all its faces are acute-angled triangles.