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–	Day 1
1	Find a way of arranging n girls and n boys around a round table for which $d_n - c_n$ is maximum, where dn is the number of girls sitting between two boys and c_n is the number of boys sitting between two girls.
2	In a cyclic quadrilateral $ABCD$ the line passing through the midpoint of AB and the intersection point of the diagonals is perpendicular to CD. Prove that either the sides AB and CD are parallel or the diagonals are perpendicular
3	Find all pairs of positive numbers (x, y) which satisfy the system of equations $x^2+y^2 = a^2+b^2$ $x^3 + y^3 = a^3 + b^3$ where a and b are given positive numbers.
–	Day 2
4	On a plane is given a finite set of points. Prove that the points can be covered by open squares $Q_1, Q_2, ..., Q_n$ such that $1 \leq \frac{N_j}{S_j} \leq 4$ for $j = 1, ..., n$, where N_j is the number of points from the set inside square Q_j and S_j is the area of Q_j.
5	Integers $x_0, x_1, ..., x_{n-1}, x_n = x_0, x_{n+1} = x_1$ satisfy the inequality $(-1)^{x_k} x_{k-1} x_{k+1} > 0$ for $k = 1, 2, ..., n$. Prove that the difference $\sum_{k=0}^{n-1} x_k - \sum_{k=0}^{n-1} \lvert x_k \rvert$ is divisible by 4.
6	Prove that the sum of dihedral angles in an arbitrary tetrahedron is greater than 2π