Tournament of Towns 2011

www.artofproblemsolving.com/community/c1081820

by parmenides51, Googlu15, sartt

–   Spring 2011

–   Junior O-Level

1   The numbers from 1 to 2010 inclusive are placed along a circle so that if we move along the circle in clockwise order, they increase and decrease alternately. Prove that the difference between some two adjacent integers is even.

2   A rectangle is divided by 10 horizontal and 10 vertical lines into 121 rectangular cells. If 111 of them have integer perimeters, prove that they all have integer perimeters.

3   Worms grow at the rate of 1 metre per hour. When they reach their maximal length of 1 metre, they stop growing. A full-grown worm may be dissected into two not necessarily equal parts. Each new worm grows at the rate of 1 metre per hour. Starting with 1 full-grown worm, can one obtain 10 full-grown worms in less than 1 hour?

4   Each diagonal of a convex quadrilateral divides it into two isosceles triangles. The two diagonals of the same quadrilateral divide it into four isosceles triangles. Must this quadrilateral be a square?

5   A dragon gave a captured knight 100 coins. Half of them are magical, but only dragon knows which are. Each day, the knight should divide the coins into two piles (not necessarily equal in size). The day when either magic coins or usual coins are spread equally between the piles, the dragon set the knight free. Can the knight guarantee himself a freedom in at most
(a) 50 days?
(b) 25 days?

–   Junior A-Level

1   Does there exist a hexagon that can be divided into four congruent triangles by a straight cut?

2   Passing through the origin of the coordinate plane are 180 lines, including the coordinate axes, which form 1 degree angles with one another at the origin. Determine the sum of the x-coordinates of the points of intersection of these lines with the line $y = 100 - x$

3   Baron Munchausen has a set of 50 coins. The mass of each is a distinct positive integer not exceeding 100, and the total mass is even. The Baron claims that it is not possible to divide

the coins into two piles with equal total mass. Can the Baron be right?

4   Given an integer $n > 1$, prove that there exist distinct positive integers a, b, c and d such that $a + b = c + d$ and $\frac{a}{b} = \frac{nc}{d}$.

5 AD and BE are altitudes of an acute triangle ABC. From D, perpendiculars are dropped to AB at G and AC at K. From E, perpendiculars are dropped to AB at F and BC at H. Prove that FG is parallel to HK and $FK = GH$.

6   Two ants crawl along the sides of the 49 squares of a $7 * 7$ board. Each ant passes through all 64 vertices exactly once and returns to its starting point. What is the smallest possible number of sides covered by both ants?

7   In every cell of a square table is a number. The sum of the largest two numbers in each row is a and the sum of the largest two numbers in each column is b. Prove that $a = b$.

–   Senior O-Level

1   The faces of a convex polyhedron are similar triangles. Prove that this polyhedron has two pairs of congruent faces.

2   same as Junior O p3

3   Along a circle are 100 white points. An integer k is given, where $2 \le k \le 50$. In each move, we choose a block of k adjacent points such that the first and the last are white, and we paint both of them black. For which values of k is it possible for us to paint all 100 points black after 50 moves?

4   Four perpendiculars are drawn from four vertices of a convex pentagon to the opposite sides. If these four lines pass through the same point, prove that the perpendicular from the fifth vertex to the opposite side also passes through this point.

5   In a country, there are 100 towns. Some pairs of towns are joined by roads. The roads do not intersect one another except meeting at towns. It is possible to go from any town to any other town by road. Prove that it is possible to pave some of the roads so that the number of paved roads at each town is odd.

–   Senior A-Level

1   same as Junior A p3

2   In the coordinate space, each of the eight vertices of a rectangular box has integer coordinates. If the volume of the solid is 2011, prove that the sides of the rectangular box are parallel to the

coordinate axes.

3   (a) Does there exist an in nite triangular beam such that two of its cross-sections are similar but not congruent triangles?
(b) Does there exist an in nite triangular beam such that two of its cross-sections are equilateral triangles of sides 1 and 2 respectively?

4   There are n red sticks and n blue sticks. The sticks of each colour have the same total length, and can be used to construct an n-gon. We wish to repaint one stick of each colour in the other colour so that the sticks of each colour can still be used to construct an n-gon. Is this always possible if
(a) $n = 3$,
(b) $n > 3$?

5   In the convex quadrilateral $ABCD$, BC is parallel to AD. Two circular arcs ω_1 and ω_3 pass through A and B and are on the same side of AB. Two circular arcs ω_2 and ω_4 pass through C and D and are on the same side of CD. The measures of $\omega_1, \omega_2, \omega_3$ and ω_4 are α, β, β and α respectively. If ω_1 and ω_2 are tangent to each other externally, prove that so are ω_3 and ω_4.

6   same as Junior A p7

7   Among a group of programmers, every two either know each other or do not know each other. Eleven of them are geniuses. Two companies hire them one at a time, alternately, and may not hire someone already hired by the other company. There are no conditions on which programmer a company may hire in the fi rst round. Thereafter, a company may only hire a programmer who knows another programmer already hired by that company. Is it possible for the company which hires second to hire ten of the geniuses, no matter what the hiring strategy of the other company may be?

–   Fall 2011

–   Junior O-Level

1 P and Q are points on the longest side AB of triangle ABC such that $AQ = AC$ and $BP = BC$. Prove that the circumcentre of triangle CPQ coincides with the incentre of triangle ABC.

2   Several guests at a round table are eating from a basket containing 2011 berries. Going in clockwise direction, each guest has eaten either twice as many berries as or six fewer berries than the next guest. Prove that not all the berries have been eaten.

3   From the 9×9 chessboard, all 16 unit squares whose row numbers and column numbers are both even have been removed. Disect the punctured board into rectangular pieces, with as few of them being unit squares as possible.

4   The vertices of a 33-gon are labelled with the integers from 1 to 33. Each edge is then labelled with the sum of the labels of its two vertices. Is it possible for the edge labels to consist of 33 consecutive numbers?

5   On a highway, a pedestrian and a cyclist were going in the same direction, while a cart and a car were coming from the opposite direction. All were travelling at different constant speeds. The cyclist caught up with the pedestrian at 10 o'clock. After a time interval, she met the cart, and after another time interval equal to the first, she met the car. After a third time interval, the car met the pedestrian, and after another time interval equal to the third, the car caught up with the cart. If the pedestrian met the car at 11 o'clock, when did he meet the cart?

–   Junior A-Level

1   An integer $N > 1$ is written on the board. Alex writes a sequence of positive integers, obtaining new integers in the following manner: he takes any divisor greater than 1 of the last number and either adds it to, or subtracts it from the number itself. Is it always (for all $N > 1$) possible for Alex to write the number 2011 at some point?

2   On side AB of triangle ABC a point P is taken such that $AP = 2PB$. It is known that $CP = 2PQ$ where Q is the midpoint of AC. Prove that ABC is a right triangle.

3   A balance and a set of pairwise different weights are given. It is known that for any pair of weights from this set put on the left pan of the balance, one can counterbalance them by one or several of the remaining weights put on the right pan. Find the least possible number of weights in the set.

4   A checkered table consists of 2012 rows and $k > 2$ columns. A marker is placed in a cell of the left-most column. Two players move the marker in turns. During each move, the player moves the marker by 1 cell to the right, up or down to a cell that had never been occupied by the marker before. The game is over when any of the players moves the marker to the right-most column. However, whether this player is to win or to lose, the players are advised only when the marker reaches the second column from the right. Can any player secure his win?

5   Given that $0 < a, b, c, d < 1$ and $abcd = (1 - a)(1 - b)(1 - c)(1 - d)$, prove that $(a + b + c + d) - (a + c)(b + d) \geq 1$

6   A car goes along a straight highway at the speed of 60 km per hour. A 100 meter long fence is standing parallel to the highway. Every second, the passenger of the car measures the angle of vision of the fence. Prove that the sum of all angles measured by him is less than 1100 degrees.

7   The vertices of a regular 45-gon are painted into three colors so that the number of vertices of each color is the same. Prove that three vertices of each color can be selected so that three triangles formed by the chosen vertices of the same color are all equal.

–   Senior O-Level

1   same as Junior O p2

2   Peter buys a lottery ticket on which he enters an n-digit number, none of the digits being 0. On the draw date, the lottery administrators will reveal an $n \times n$ table, each cell containing one of the digits from 1 to 9. A ticket wins a prize if it does not match any row or column of this table, read in either direction. Peter wants to bribe the administrators to reveal the digits on some cells chosen by Peter, so that Peter can guarantee to have a winning ticket. What is the minimum number of digits Peter has to know?

3   In a convex quadrilateral $ABCD, AB = 10, BC = 14, CD = 11$ and $DA = 5$. Determine the angle between its diagonals.

4   Positive integers $a < b < c$ are such that $b + a$ is a multiple of $b - a$ and $c + b$ is a multiple of $c - b$. If a is a 2011-digit number and b is a 2012-digit number, exactly how many digits does c have?

5   In the plane are 10 lines in general position, which means that no 2 are parallel and no 3 are concurrent. Where 2 lines intersect, we measure the smaller of the two angles formed between them. What is the maximum value of the sum of the measures of these 45 angles?

–   Senior A-Level

1   Pete has marked several (three or more) points in the plane such that all distances between them are different. A pair of marked points A, B will be called unusual if A is the furthest marked point from B, and B is the nearest marked point to A (apart from A itself). What is the largest possible number of unusual pairs that Pete can obtain?

2   same as Junior A p5

3   In triangle ABC, points A_1, B_1, C_1 are bases of altitudes from vertices A, B, C, and points C_A, C_B are the projections of C_1 to AC and BC respectively. Prove that line $C_A C_B$ bisects the segments $C_1 A_1$ and $C_1 B_1$.

4   Does there exist a convex N-gon such that all its sides are equal and all vertices belong to the parabola $y = x^2$ for
a) $N = 2011$
b) $N = 2012$?

5   We will call a positive integer *good* if all its digits are nonzero. A good integer will be called *special* if it has at least k digits and their values strictly increase from left to right. Let a good integer be given. At each move, one may either add some special integer to its digital expression from the left or from the right, or insert a special integer between any two its digits, or remove a special number from its digital expression. What is the largest k such that any good integer can be turned into any other good integer by such moves?

6   Prove that the integer $1^1 + 3^3 + 5^5 + .. + (2^n - 1)^{2^n - 1}$ is a multiple of 2^n but not a multiple of 2^{n+1}.

7 100 red points divide a blue circle into 100 arcs such that their lengths are all positive integers from 1 to 100 in an arbitrary order. Prove that there exist two perpendicular chords with red endpoints.

–   Oral Round

1   There are n coins in a row. Two players take turns picking a coin and flipping it. The location of the heads and tails should not repeat. Loses the one who can not make a move. Which of player can always win, no matter how his opponent plays?

2 49 natural numbers are written on the board. All their pairwise sums are different. Prove that the largest of the numbers is greater than 600.

    49 .   ., 600

3   Three pairwise intersecting rays are given. At some point in time not on every ray from its beginning a point begins to move with speed. It is known that these three points form a triangle at any time, and the center of the circumscribed circle of this the triangle also moves uniformly and on a straight line. Is it true, that all these triangles are similar to each other?

4   A subset of a student group is called an *ideal company* if
    1) in this subset, all girls are liked by all young men,
    2) no one can be added to this subset without violating condition 1.
    In a certain group, 9 female students and 15 students study. Warden of the group made a list of all kinds of ideal companies in this group. What is the largest number of companies on this list?

5   Find all positive integers a, b such that b^{619} divides $a^{1000} + 1$ and a^{619} divides $b^{1000} + 1$.

6   On the plane there are centrally symmetric convex polygon with area 1 and two his copies (each obtained from a polygon by some parallel transfer). It is known that no point of the plane is not covered by the three polygons at once. Prove that the total area covered by polygons, at least 2.