

**Shortlisted Problems to the Junior Olympiad of Malaysia**
[www.artofproblemsolving.com/community/c108392](http://www.artofproblemsolving.com/community/c108392)

by zschess

**A1** Let  $a, b, c$  be the side lengths of a triangle. Prove that

$$\sum_{cyc} \frac{(a^2 + b^2)(a + c)}{b} \geq 2(a^2 + b^2 + c^2)$$

**A2** Let  $a, b, c$  be positive real numbers greater or equal to 3. Prove that

$$3(abc + b + 2c) \geq 2(ab + 2ac + 3bc)$$

and determine all equality cases.

**A3** Let  $a, b, c$  be positive real numbers less than or equal to  $\sqrt{2}$  such that  $abc = 2$ , prove that

$$\sqrt{2} \sum_{cyc} \frac{ab + 3c}{3ab + c} \geq a + b + c$$

**A4** Suppose  $2015 = a_1 < a_2 < a_3 < \dots < a_k$  be a finite sequence of positive integers, and for all  $m, n \in \mathbb{N}$  and  $1 \leq m, n \leq k$ ,

$$a_m + a_n \geq a_{m+n} + |m - n|$$

Determine the largest possible value  $k$  can obtain.

**A5** Let  $a, b, c$  be the side length of a triangle, with  $ab + bc + ca = 18$  and  $a, b, c > 1$ . Prove that

$$\sum_{cyc} \frac{1}{(a-1)^3} > \frac{1}{a+b+c-3}$$

**A6** Let  $(a_n)_{n \geq 0}$  and  $(b_n)_{n \geq 0}$  be two sequences with arbitrary real values  $a_0, a_1, b_0, b_1$ . For  $n \geq 1$ , let  $a_{n+1}, b_{n+1}$  be defined in this way:

$$a_{n+1} = \frac{b_{n-1} + b_n}{2}, b_{n+1} = \frac{a_{n-1} + a_n}{2}$$

Prove that for any constant  $c > 0$  there exists a positive integer  $N$  s.t. for all  $n > N$ ,  $|a_n - b_n| < c$ .

**A7** Given positive reals  $a, b, c$  that satisfy  $a + b + c = 1$ , show that

$$\sum_{cyc} \frac{a^3 + bc}{a^2 + bc} \geq 2$$

**A8** Let  $a_1, a_2, \dots, a_{2015}$  be 2015-tuples of positive integers (not necessary distinct) and let  $k$  be a positive integers. Denote  $f(i) = a_i + \frac{a_1 a_2 \cdots a_{2015}}{a_i}$ .

a) Prove that if  $k = 2015^{2015}$ , there exist  $a_1, a_2, \dots, a_{2015}$  such that  $f(i) = k$  for all  $1 \leq i \leq 2015$ .

b) Find the maximum  $k_0$  so that for  $k \leq k_0$ , there are no  $k$  such that there are at least 2 different 2015-tuples which fulfill the above condition.

**A9** Let  $2n$  positive reals  $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$  satisfy  $a_{i+1} \geq 2a_i$  and  $b_{i+1} \leq b_i$  for  $1 \leq i \leq n-1$ . Find the least constant  $C$  that satisfy:

$$\sum_{i=1}^n \frac{a_i}{b_i} \geq \frac{C(a_1 + a_2 + \cdots + a_n)}{b_1 + b_2 + \cdots + b_n}$$

and determine all equality case with that constant  $C$ .

**C1** Baron and Peter are playing a game. They are given a simple finite graph  $G$  with  $n \geq 3$  vertex and  $k$  edges that connects the vertices. First Peter labels two vertices A and B, and places a counter at A. Baron starts first. A move for Baron is move the counter along an edge. Peter's move is to remove an edge from the graph. Baron wins if he reaches B, otherwise Peter wins.

Given the value of  $n$ , what is the largest  $k$  so that Peter can always win?

**C2** Cauchy the magician has a new card trick. He takes a standard deck(which consists of 52 cards with 13 denominations in each 4 suits) and let Schwartz to shuffle randomly. Schwartz is told to take  $m$  cards not more than  $\frac{1}{3}$  form the top of the deck. Then, Cauchy take 18 cards one by one from the top of the remaining deck and show it to Schwartz with the second card is placed in front of the first card (from Schwartz view) and so on. He ask Schwartz to memorize the  $m$ -th card when showing the cards. Let it be  $C_1$ . After that, Cauchy places the 18 cards and the  $m$  cards on the bottom of the deck with the  $m$  cards are placed lower than the 18 cards. Now, Cauchy distributes and flip the cards on the table from the top of the deck while shouting the numbers 10 until 1 with the following operation:

a) When a card flipped has the number which is same as the number shouted by Cauchy, stop the distribution and continue with another set.

b) When 10 cards are flipped and none of the cards flipped has the number which is same as the number shouted by Cauchy, take a card from the top of the deck and place it on top of the

set with backside(the site which has no value) facing up. Then continue with another set.

Cauchy stops when 3 sets of cards are placed. Then, he adds up all the numbers on top of each sets of cards( backside is consider 0 ). Let  $k$  be the sum. He placed another  $k$  cards to the table from the top of the remaining deck. Finally, he shows the first card on top of the remaining deck to Schwartz. Let it be  $C_2$ .

Show that  $C_1 = C_2$ .

**C3** Let  $n \geq 2$  be a positive integer and  $S = \{1, 2, \dots, n\}$ . Let two functions  $f : S \rightarrow \{1, -1\}$  and  $g : S \rightarrow S$  satisfy:

i)  $f(x)f(y) = f(x + y), \forall x, y \in S$

ii)  $f(g(x)) = f(x), \forall x \in S$

iii)  $f(x + n) = f(x), \forall x \in S$

iv)  $g$  is bijective.

Find the number of pair of such functions  $(f, g)$  for every  $n$ .

**C4** Nikees has a set  $S$  of  $n$  points on a plane and decides to colour them. All  $\binom{n}{2}$  line segments are drawn and they have distinct lengths. Find the maximum number of colours that are used at least once, given that:

(a) For each point  $P$ , the two endpoints of the longest line segment connecting  $P$  must be of the same colour.

(b) For each point  $P$ , the two endpoints of the shortest line segment connecting  $P$  must be of the same colour.

**C5** Let  $G$  be a simple connected graph. Each edge has two phases, which is either blue or red. Each vertex are switches that change the colour of every edge that connects the vertex. All edges are initially red. Find all ordered pairs  $(n, k), n \geq 3$ , such that:

a) For all graph  $G$  with  $n$  vertex and  $k$  edges, it is always possible to perform a series of switching process so that all edges are eventually blue.

b) There exist a graph  $G$  with  $n$  vertex and  $k$  edges and it is possible to perform a series of switching process so that all edges are eventually blue.

**C6** In a massive school which has  $m$  students, and each student took at least one subject. Let  $p$  be an odd prime. Given that:

- (i) each student took at most  $p + 1$  subjects.
- (ii) each subject is taken by at most  $p$  students.
- (iii) any pair of students has at least 1 subject in common.

Find the maximum possible value of  $m$ .

- C7** Navi and Ozna are playing a game where Ozna starts first and the two take turn making moves. A positive integer is written on the board. A move is to (i) subtract any positive integer at most 2015 from it or (ii) given that the integer on the board is divisible by 2014, divide by 2014. The first person to make the integer 0 wins. To make Navi's condition worse, Ozna gets to pick integers  $a$  and  $b$ ,  $a \geq 2015$  such that all numbers of the form  $an + b$  will not be the starting integer, where  $n$  is any positive integer.

Find the minimum number of starting integer where Navi wins.

- C8** Let  $a$  be a permutation on  $\{0, 1, \dots, 2015\}$  and  $b, c$  are also permutations on  $\{1, 2, \dots, 2015\}$ . For all  $x \in \{1, 2, \dots, 2015\}$ , the following conditions are satisfied:

- (i)  $a(x) - a(x - 1) \neq 1$ ,
- (ii) if  $b(x) \neq x$ , then  $c(x) = x$ ,

Prove that the number of  $a$ 's is equal to the number of ordered pairs of  $(b, c)$ .

- G1** Given a triangle  $ABC$ , and let  $E$  and  $F$  be the feet of altitudes from vertices  $B$  and  $C$  to the opposite sides. Denote  $O$  and  $H$  be the circumcenter and orthocenter of triangle  $ABC$ . Given that  $FA = FC$ , prove that  $OEHF$  is a parallelogram.

- G2** Let  $ABC$  be a triangle, and let  $M$  be midpoint of  $BC$ . Let  $I_b$  and  $I_c$  be incenters of  $AMB$  and  $AMC$ . Prove that the second intersection of circumcircles of  $ABI_b$  and  $ACI_c$  distinct from  $A$  lies on line  $AM$ .

- G3** Let  $ABC$  a triangle. Let  $D$  on  $AB$  and  $E$  on  $AC$  such that  $DE \parallel BC$ . Let line  $DE$  intersect circumcircle of  $ABC$  at two distinct points  $F$  and  $G$  so that line segments  $BF$  and  $CG$  intersect at  $P$ . Let circumcircle of  $GDP$  and  $FEP$  intersect again at  $Q$ . Prove that  $A, P, Q$  are collinear.

- G4** Let  $ABC$  be a triangle and let  $AD, BE, CF$  be cevians of the triangle which are concurrent at  $G$ . Prove that if  $CF \cdot BE \geq AF \cdot EC + AE \cdot BF + BC \cdot FE$  then  $AG \leq GD$ .

- G5** Let  $ABCD$  be a convex quadrilateral. Let angle bisectors of  $\angle B$  and  $\angle C$  intersect at  $E$ . Let  $AB$  intersect  $CD$  at  $F$ .

Prove that if  $AB + CD = BC$ , then  $A, D, E, F$  is cyclic.

**G6** Let  $ABC$  be a triangle. Let  $\omega_1$  be circle tangent to  $BC$  at  $B$  and passes through  $A$ . Let  $\omega_2$  be circle tangent to  $BC$  at  $C$  and passes through  $A$ . Let  $\omega_1$  and  $\omega_2$  intersect again at  $P \neq A$ . Let  $\omega_1$  intersect  $AC$  again at  $E \neq A$ , and let  $\omega_2$  intersect  $AB$  again at  $F \neq A$ . Let  $R$  be the reflection of  $A$  about  $BC$ , Prove that lines  $BE, CF, PR$  are concurrent.

**G7** Let  $ABC$  be an acute triangle. Let  $H_A, H_B, H_C$  be points on  $BC, AC, AB$  respectively such that  $AH_A \perp BC, BH_B \perp AC, CH_C \perp AB$ . Let the circumcircles  $AH_BH_C, BH_AH_C, CH_AH_B$  be  $\omega_A, \omega_B, \omega_C$  with circumcenters  $O_A, O_B, O_C$  respectively and define  $O_{AB} \cap \omega_B = P_{AB} \neq B$ . Define  $P_{AC}, P_{BA}, P_{BC}, P_{CA}, P_{CB}$  similarly. Define circles  $\omega_{AB}, \omega_{AC}$  to be  $O_A P_{AB} H_C, O_A P_{AC} H_B$  respectively. Define circles  $\omega_{BA}, \omega_{BC}, \omega_{CA}, \omega_{CB}$  similarly.

Prove that there are 6 pairs of tangent circles in the 6 circles of the form  $\omega_{xy}$ .

**G8** Let  $ABCDE$  be a convex pentagon such that  $BC$  and  $DE$  are tangent to the circumcircle of  $ACD$ . Prove that if the circumcircles of  $ABC$  and  $ADE$  intersect at the midpoint of  $CD$ , then the circumcircles  $ABE$  and  $ACD$  are tangent to each other.

**N1** Prove that there exists an infinite sequence of positive integers  $a_1, a_2, \dots$  such that for all positive integers  $i$ ,

i)  $a_{i+1}$  is divisible by  $a_i$ .

ii)  $a_i$  is not divisible by 3.

iii)  $a_i$  is divisible by  $2^{i+2}$  but not  $2^{i+3}$ .

iv)  $6a_i + 1$  is a prime power.

v)  $a_i$  can be written as the sum of the two perfect squares.

**N2** Let  $\mathbb{A} \subset \mathbb{N}$  such that all elements in  $\mathbb{A}$  can be representable in the form of  $x^2 + 2y^2$ ,  $x, y \in \mathbb{N}$ , and  $x > y$ . Let  $\mathbb{B} \subset \mathbb{N}$  such that all elements in  $\mathbb{B}$  can be representable in the form of  $\frac{a^3 + b^3 + c^3}{a + b + c}$ ,  $a, b, c \in \mathbb{N}$ , and  $a, b, c$  are distinct.

a) Prove that  $\mathbb{A} \subset \mathbb{B}$ .

b) Prove that there exist infinitely many positive integers  $n$  satisfy  $n \in \mathbb{B}$  and  $n \notin \mathbb{A}$

**N3** Given a natural number  $n \geq 3$ , determine all strictly increasing sequences  $a_1 < a_2 < \dots < a_n$

such that  $\gcd(a_1, a_2) = 1$  and for any pair of natural numbers  $(k, m)$  satisfy  $n \geq m \geq 3, m \geq k$ ,

$$\frac{a_1 + a_2 + \cdots + a_m}{a_k}$$

is a positive integer.

**N4** Determine all triplet of non-negative integers  $(x, y, z)$  satisfy

$$2^x 3^y + 1 = 7^z$$

**N5** Let  $a, b, c$  be pairwise coprime positive integers. Find all positive integer values of

$$\frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b}$$

**N6** Let  $p_i$  denote the  $i$ -th prime number. Let  $n = \lfloor \alpha^{2015} \rfloor$ , where  $\alpha$  is a positive real number such that  $2 < \alpha < 2.7$ . Prove that

$$\sum_{2 \leq p_i \leq p_j \leq n} \frac{1}{p_i p_j} < 2017$$

**N7** Find all functions  $f : \mathbb{N} \rightarrow \mathbb{N}_0$  satisfy the following conditions:

i)  $f(ab) = f(a) + f(b) - f(\gcd(a, b)), \forall a, b \in \mathbb{N}$

ii) For all primes  $p$  and natural numbers  $a, f(a) \geq f(ap) \Rightarrow f(a) + f(p) \geq f(a)f(p) + 1$

**N8** Set  $p \geq 5$  be a prime number and  $n$  be a natural number. Let  $f$  be a function  $f : \mathbb{Z}_{\neq 0} \rightarrow \mathbb{N}_0$  satisfy the following conditions:

i) For all sequences of integers satisfy  $a_i \notin \{0, 1\}$ , and  $p \nmid a_i - 1, \forall 1 \leq i \leq p - 2$ ,

$$\sum_{i=1}^{p-2} f(a_i) = f(a_1 a_2 \cdots a_{p-2})$$

ii) For all coprime integers  $a$  and  $b, a \equiv b \pmod{p} \Rightarrow f(a) = f(b)$

iii) There exist  $k \in \mathbb{Z}_{\neq 0}$  that satisfy  $f(k) = n$

Prove that the number of such functions is  $d(n)$ , where  $d(n)$  denotes the number of divisors of  $n$ .