Problems from the 2020 BAMO-8 and BAMO-12 exams

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A A trapezoid is divided into seven strips of equal width as shown. What fraction of the trapezoids area is shaded?

B Four friends, Anna, Bob, Celia, and David, exchanged some money. For any two of these friends, exactly one gave money to the other. For example, Celia could have given some money to David but then David would not have given money to Celia. In the end, each person broke even (meaning that no one made or lost any money).

(a) Is it possible that the amounts of money given were $10, 20, 30, 40, 50, 60$?
(b) Is it possible that the amounts of money given were $20, 30, 40, 50, 60, 70$?

For each part, if your answer is yes, show that the situation is possible by describing who could have given what amounts to whom. If your answer is no, prove that the situation is not possible.

C/1 Find all real numbers x that satisfy the equation

$$\frac{x-2020}{1}+\frac{x-2019}{2}+\cdots+\frac{x-2000}{21}=\frac{x-1}{2020}+\frac{x-2}{2019}+\cdots+\frac{x-21}{2000},$$

and simplify your answer(s) as much as possible. Justify your solution.

D/2 Consider a sheet of paper in the shape of an equilateral triangle creased along the dashed lines as in the figure below on the left. Folding over each of the three corners along the dashed lines creates a new object which is uniformly four layers thick, as in the figure below on the right. The number in each region indicates that region's thickness (in layers of paper).
We have just seen one example of how a plane figure can be folded into an object with a uniform thickness. This problem asks you do produce several other examples. In each case, you may fold along any lines. The different parts that are folded may or may not be congruent. Assume that paper may be folded any number of times without tearing or becoming too thick to fold. In needed, you can use any of the following tools:

- A magic ruler with which you can draw a line through any two given points and you can split any segment into many equal parts as you wish; and
- A right triangle tool with which you can drop perpendiculars from points to lines and erect perpendiculars to lines from points on them.

E/3 The integer 202020 is a multiple of 91. For each positive integer n, show how n additional 2's may be inserted into the digits of 202020 such that the resulting $(n+6)$-digit number is also a multiple of 91.

For example, a possible way to do this when $n = 5$ is 22020220222 (the inserted 2's are underlined).

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4 Consider $\triangle ABC$. Choose a point M on side BC and let O be the center of the circle passing through the vertices of $\triangle ABM$. Let k be the circle that passes through A and M and whose center lies on BC. Let line MO intersect K again in point K. Prove that the line BK is the same for any point M on segment BC, so long as all of these constructions are well-defined.

Proposed by Evan Chen

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5 Let S be a set of $a + b + 3$ points on a sphere, where a, b are nonnegative integers and no four points of S are coplanar. Determine how many planes pass through three points of S and separate the remaining points into a points on one side of the plane and b points on the other side.
