12th RMM 2020

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by parmenides51, pinetree1, magicarrow, Ankoganit

–	Day 1
1	Let ABC be a triangle with a right angle at C. Let I be the incentre of triangle ABC, and let D be the foot of the altitude from C to AB. The incircle ω of triangle ABC is tangent to sides BC, CA, and AB at A_1, B_1, and C_1, respectively. Let E and F be the reflections of C in lines C_1A_1 and C_1B_1, respectively. Let K and L be the reflections of D in lines C_1A_1 and C_1B_1, respectively. Prove that the circumcircles of triangles A_1EI, B_1FI, and C_1KL have a common point.
2	Let $N \geq 2$ be an integer, and let $\mathbf{a} = (a_1, \ldots, a_N)$ and $\mathbf{b} = (b_1, \ldots b_N)$ be sequences of non-negative integers. For each integer $i \notin \{1, \ldots, N\}$, let $a_i = a_k$ and $b_i = b_k$, where $k \in \{1, \ldots, N\}$ is the integer such that $i - k$ is divisible by n. We say $\mathbf{a}$ is $\mathbf{b}$-*harmonic* if each a_i equals the following arithmetic mean: $$a_i = \frac{1}{2b_i+1} \sum_{s=-b_i}^{b_i} a_{i+s}.$$ Suppose that neither $\mathbf{a}$ nor $\mathbf{b}$ is a constant sequence, and that both $\mathbf{a}$ is $\mathbf{b}$-*harmonic* and $\mathbf{b}$ is $\mathbf{a}$-*harmonic*. Prove that at least $N+1$ of the numbers $a_1, \ldots, a_N, b_1, \ldots, b_N$ are zero.
3	Let $n \geq 3$ be an integer. In a country there are n airports and n airlines operating two-way flights. For each airline, there is an odd integer $m \geq 3$, and m distinct airports $c_1, \ldots, c_m$, where the flights offered by the airline are exactly those between the following pairs of airports: c_1 and c_2; c_2 and c_3; $\ldots$; c_{m-1} and c_m; c_m and c_1. Prove that there is a closed route consisting of an odd number of flights where no two flights are operated by the same airline.
–	Day 2
4	Let $\mathbb{N}$ be the set of all positive integers. A subset A of $\mathbb{N}$ is *sum-free* if, whenever x and y are (not necessarily distinct) members of A, their sum $x+y$ does not belong to A. Determine all surjective functions $f : \mathbb{N} \to \mathbb{N}$ such that, for each sum-free subset A of $\mathbb{N}$, the image $\{f(a) : a \in A\}$ is also sum-free. [i]Note: a function $f : \mathbb{N} \to \mathbb{N}$ is surjective if, for every positive integer n, there exists a positive integer m such that $f(m) = n$.[/i]

5 A *lattice point* in the Cartesian plane is a point whose coordinates are both integers. A *lattice polygon* is a polygon all of whose vertices are lattice points.

Let Γ be a convex lattice polygon. Prove that Γ is contained in a convex lattice polygon Ω such that the vertices of Γ all lie on the boundary of Ω, and exactly one vertex of Ω is not a vertex of Γ.

6 For each integer $n \geq 2$, let $F(n)$ denote the greatest prime factor of n. A *strange pair* is a pair of distinct primes p and q such that there is no integer $n \geq 2$ for which $F(n)F(n+1) = pq$.

Prove that there exist infinitely many strange pairs.