Kurschak Competition 2011

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by Ti-Ci

1 Let $a_1, a_2, ...$ be an infinite sequence of positive integers such that for any $k, \ell \in \mathbb{Z}_+$, $a_{k+\ell}$ is divisible by $\gcd(a_k, a_\ell)$. Prove that for any integers $1 \leqslant k \leqslant n$, $a_n a_{n-1} \dots a_{n-k+1}$ is divisible by $a_k a_{k-1} \dots a_1$.

2 Let n be a positive integer. Denote by $a(n)$ the ways of expression $n = x_1 + x_2 + \dots$ where $x_1 \leqslant x_2 \leqslant \dots$ are positive integers and $x_i + 1$ is a power of 2 for each i. Denote by $b(n)$ the ways of expression $n = y_1 + y_2 + \dots$ where y_i is a positive integer and $2y_i \leqslant y_{i+1}$ for each i.
Prove that $a(n) = b(n)$.

3 Given $2n$ points and $3n$ lines on the plane. Prove that there is a point P on the plane such that the sum of the distances of P to the $3n$ lines is less than the sum of the distances of P to the $2n$ points.