Final Round - Switzerland 2020

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– Day 1

1 Let $\mathbb{N}$ be the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for every $m, n \in \mathbb{N}$,

$$f(m) + f(n) \mid m + n.$$

2 Let ABC be an acute triangle. Let M_A, M_B and M_C be the midpoints of sides BC, CA, respectively AB. Let M'_A, M'_B and M'_C be the the midpoints of the arcs BC, CA and AB respectively of the circumscriberd circle of triangle ABC. Let P_A be the intersection of the straight line M_BM_C and the perpendicular to $M'_BM'_C$ through A. Define P_B and P_C similarly. Show that the straight line M_AP_A, M_BP_B and M_CP_C intersect at one point.

3 We are given n distinct rectangles in the plane. Prove that between the $4n$ interior angles formed by these rectangles at least $4\sqrt{n}$ are distinct.

4 Let φ denote the Euler phi-function. Prove that for every positive integer n

$$2^{n(n+1)} | 32 \cdot \varphi\left(2^{2^n} - 1\right).$$

– Day 2

5 Find all the positive integers a, b, c such that

$$a! \cdot b! = a! + b! + c!$$

6 Let $n \geq 2$ be an integer. Consider the following game: Initially, k stones are distributed among the n^2 squares of an $n \times n$ chessboard. A move consists of choosing a square containing at least as many stones as the number of its adjacent squares (two squares are adjacent if they share a common edge) and moving one stone from this square to each of its adjacent squares. Determine all positive integers k such that:
(a) There is an initial configuration with k stones such that no move is possible.
(b) There is an initial configuration with k stones such that an infinite sequence of moves is possible.

7 Let $ABCD$ be an isosceles trapezoid with bases $AD > BC$. Let X be the intersection of the bisectors of $\angle BAC$ and BC. Let E be the intersection ofDB with the parallel to the bisector of $\angle CBD$ through X and let F be the intersection of DC with the parallel to the bisector of $\angle DCB$ through X. Show that quadrilateral $AEFD$ is cyclic.

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8 Let n be a positive integer . Let $x_1 \le x_2 \le \cdots \le x_m$ be a sequence of real numbers such that $x_1 + x_2 + \cdots + x_n = 0$ and $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Prove that $x_1 x_n \leqslant -\frac{1}{n}$.

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