Sharygin Geometry Olympiad 2020

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–	First (Correspondence) Round
1	Let ABC be a triangle with $\angle C = 90^\circ$, and A_0, B_0, C_0 be the mid-points of sides BC, CA, AB respectively. Two regular triangles AB_0C_1 and BA_0C_2 are constructed outside ABC. Find the angle $C_0C_1C_2$.
2	Let $ABCD$ be a cyclic quadrilateral. A circle passing through A and B meets AC and BD at points E and F respectively. The lines AF and BC meet at point P, and the lines BE and AD meet at point Q. Prove that PQ is parallel to CD.
3	Let ABC be a triangle with $\angle C = 90^\circ$, and D be a point outside ABC, such that $\angle ADC = \angle BAC$. The segments CD and AB meet at point E. It is known that the distance from E to AC is equal to the circumradius of triangle ADE. Find the angles of triangle ABC.
4	Let $ABCD$ be an isosceles trapezoid with bases AB and CD. Prove that the centroid of triangle ABD lies on CF where F is the projection of D to AB.
5	Let BB_1, CC_1 be the altitudes of triangle ABC, and AD be the diameter of its circumcircle. The lines BB_1 and DC_1 meet at point E, the lines CC_1 and DB_1 meet at point F. Prove that $\angle CAE = \angle BAF$.
6	Circles ω_1 and ω_2 meet at point P, Q. Let O be the common point of external tangents of ω_1 and ω_2. A line passing through O meets ω_1 and ω_2 at points A, B located on the same side with respect to line segment PQ.The line PA meets ω_2 for the second time at C and the line QB meets ω_1 for the second time at D. Prove that $O-C-D$ are collinear.
7	Prove that the medial lines of triangle ABC meets the sides of triangle formed by its excenters at six concyclic points.
8	Two circles meeting at points P and R are given. Let ℓ_1, ℓ_2 be two lines passing through P. The line ℓ_1 meets the circles for the second time at points A_1 and B_1. The tangents at these points to the circumcircle of triangle A_1RB_1 meet at point C_1. The line C_1R meets A_1B_1 at point D_1. Points A_2, B_2, C_2, D_2 are defined similarly. Prove that the circles D_1D_2P and C_1C_2R touch.
9	The vertex A, center O and Euler line ℓ of a triangle ABC is given. It is known that ℓ intersects AB, AC at two points equidistant from A. Restore the triangle.

10 Given are a closed broken line $A_1A_2\dots A_n$ and a circle ω which touches each of lines $A_1A_2, A_2A_3, \dots, A_nA_1$. Call the link *good*, if it touches ω, and *bad* otherwise (i.e. if the extension of this link touches ω). Prove that the number of bad links is even.

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11 Let ABC be a triangle with $\angle A = 60^\circ$, AD be its bisector, and PDQ be a regular triangle with altitude DA. The lines PB and QC meet at point K. Prove that AK is a symmedian of ABC.

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12 Let H be the orthocenter of a nonisosceles triangle ABC. The bisector of angle BHC meets AB and AC at points P and Q respectively. The perpendiculars to AB and AC from P and Q meet at K. Prove that KH bisects the segment BC.

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13 Let I be the incenter of triangle ABC. The excircle with center I_A touches the side BC at point A'. The line l passing through I and perpendicular to BI meets I_AA' at point K lying on the medial line parallel to BC. Prove that $\angle B \leq 60^\circ$.

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14 A non-isosceles triangle is given. Prove that one of the circles touching internally its incircle and circumcircle and externally one of its excircles passes through a vertex of the triangle.

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15 A circle passing through the vertices B and D of quadrilateral $ABCD$ meets AB, BC, CD, and DA at points K, L, M, and N respectively. A circle passing through K and M meets AC at P and Q. Prove that L, N, P, and Q are concyclic.

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16 Cevians AP and AQ of a triangle ABC are symmetric with respect to its bisector. Let X, Y be the projections of B to AP and AQ respectively, and N, M be the projections of C to AP and AQ respectively. Prove that XM and NY meet on BC.

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17 Chords A_1A_2 and B_1B_2 meet at point D. Suppose D' is the inversion image of D and the line A_1B_1 meets the perpendicular bisector to DD' at a point C. Prove that $CD \parallel A_2B_2$.

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18 Bisectors AA_1, BB_1, and CC_1 of triangle ABC meet at point I. The perpendicular bisector to BB_1 meets AA_1, CC_1 at points A_0, C_0 respectively. Prove that the circumcircles of triangles A_0IC_0 and ABC touch.

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19 Quadrilateral $ABCD$ is such that $AB \perp CD$ and $AD \perp BC$. Prove that there exist a point such that the distances from it to the sidelines are proportional to the lengths of the corresponding sides.

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20 The line touching the incircle of triangle ABC and parallel to BC meets the external bisector of angle A at point X. Let Y be the midpoint of arc BAC of the circumcircle. Prove that the angle XIY is right.

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21 The diagonals of bicentric quadrilateral $ABCD$ meet at point L. Given are three segments equal to AL, BL, CL. Restore the quadrilateral using a compass and a ruler.

22 Let Ω be the circumcircle of cyclic quadrilateral $ABCD$. Consider such pairs of points P, Q of diagonal AC that the rays BP and BQ are symmetric with respect the bisector of angle B. Find the locus of circumcenters of triangles PDQ.

23 A non-self-intersecting polygon is nearly convex if precisely one of its interior angles is greater than 180°.

One million distinct points lie in the plane in such a way that no three of them are collinear. We would like to construct a nearly convex one-million-gon whose vertices are precisely the one million given points. Is it possible that there exist precisely ten such polygons?

24 Let I be the incenter of a tetrahedron $ABCD$, and J be the center of the exsphere touching the face BCD containing three remaining faces (outside these faces). The segment IJ meets the circumsphere of the tetrahedron at point K. Which of two segments IJ and JK is longer?