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1 We consider the two sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ of integers, which are given by $a_0 = b_0 = 2$ and $a_1 = b_1 = 14$ and for $n \geq 2$ they are defined as $a_n = 14a_{n-1} + a_{n-2}$, $b_n = 6b_{n-1} - b_{n-2}$. Determine whether there are infinite numbers that occur in both sequences

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2 Let ABC be a triangle and I its incenter. The circle passing through A, C and I intersect the line BC for second time at point X. The circle passing through B, C and I intersects the line AC for second time at point Y. Show that the segments AY and BX have equal length.

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3 Let $n \geq 2$ be an integer. Ariane and Brnice play a game on the number of the residue classes modulo n. At the beginning there is the residue class 1 on each piece of paper. It is the turn of the player whose turn it is to replace the current residue class x with either $x + 1$ or by $2x$. The two players take turns, with Ariane starting.
Ariane wins if the residue class 0 is reached during the game. Brnice wins if she can prevent that permanently.
Depending on n, determine which of the two has a winning strategy.

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4 Find all pairs (a, b) of real numbers such that $a \cdot \lfloor b \cdot n \rfloor = b \cdot \lfloor a \cdot n \rfloor$ applies to all positive integers$n$. (For a real number x, $\lfloor x \rfloor$ denotes the largest integer that is less than or equal to x.)

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