Moldova Team Selection Test 2020

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– Day 1

1 All members of geometrical progression $(b_n)_{n\geq 1}$ are members of some arithmetical progression. It is known that b_1 is an integer. Prove that all members of this geometrical progression are integers. (progression is infinite)

2 Show that for any positive real numbers a, b, c the following inequality takes place $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{a+b+c}{\sqrt{a^2+b^2+c^2}}\geq 3+\sqrt{3}$

3 Let n, $(n\geq 3)$ be a positive integer and the set A=$1,2,...,n$. All the elements of A are randomly arranged in a sequence $(a_1,a_2,...,a_n)$. The pair (a_i,a_j) forms an $inversion$ if $1\leq i\leq j\leq n$ and $a_i>a_j$. In how many different ways all the elements of the set A can be arranged in a sequence that contains exactly 3 inversions?

4 Let ΔABC be an acute triangle and H its orthocenter. B_1 and C_1 are the feet of heights from B and C, M is the midpoint of AH. Point K is on the segment B_1C_1, but isn't on line AH. Line AK intersects the lines MB_1 and MC_1 in E and F, the lines BE and CF intersect at N. Prove that K is the orthocenter of ΔNBC.

– Day 2

5 Let n be a natural number. Find all solutions x of the system of equations

$$\begin{cases} sinx+cosx=\frac{\sqrt{n}}{2} \\ tg\frac{x}{2}=\frac{\sqrt{n}-2}{3} \end{cases}$$

On interval $\left[0,\frac{\pi}{4}\right)$.

6 Let n, $(n\geq 3)$ be a positive integer and the polynomial $f(x)=(1+x)\cdot(1+2x)\cdot(1+3x)\cdot...\cdot(1+nx)$ $=a_0+a_1\cdot x+a_2\cdot x^2+a_3\cdot x^3+...+a_n\cdot x^n$. Show that the number a_3 divides the number $k=C_{n+1}^2\cdot(2\cdot C_n^2\cdot C_{n+1}^2-3\cdot a_2)$.

7 Show that for any positive real numbers a, b, c the following inequality takes place

$$\frac{a}{\sqrt{7a^2+b^2+c^2}}+\frac{b}{\sqrt{a^2+7b^2+c^2}}+\frac{c}{\sqrt{a^2+b^2+7c^2}}\leq 1.$$

8 In ΔABC the angles ABC and ACB are acute. Let M be the midpoint of AB. Point D is on the half-line $(CB$ such that $B \in (CD)$ and $\angle DAB = \angle BCM$. Perpendicular from B to line CD intersects the line bisector of AB in E. Prove that DE and AC are perpendicular.

– Day 3

9 Let ΔABC be an acute triangle and Ω its circumscribed circle, with diameter AP. Points E and F are the orthogonal projections from B on AC and AP, points M and N are the midpoints of segments EF and CP. Prove that $\angle BMN = 90$.

10 Let n be a positive integer. Positive numbers a, b, c satisfy $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. Find the greatest possible value of

$$E(a,b,c) = \frac{a^n}{a^{2n+1} + b^{2n} \cdot c + b \cdot c^{2n}} + \frac{b^n}{b^{2n+1} + c^{2n} \cdot a + c \cdot a^{2n}} + \frac{c^n}{c^{2n+1} + a^{2n} \cdot b + a \cdot b^{2n}}$$

11 Find all functions $f : [-1, 1] \to \mathbb{R}$, which satisfy

$$f(\sin x) + f(\cos x) = 2020$$

for any real number x.

12 In a chess tournament each player played one match with every other player. It is known that all players have different scores. The player who is on the last place got k points. What is the smallest number of wins that the first placed player got? (For the win 1 point is given, for loss 0 and for a draw both players get $0,5$ points.)