geometry problems from 3rd / final round from Brazilian National Math Olympiads, level 2

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1998.2 Let ABC be a triangle. D is the midpoint of AB, E is a point on the side BC such that $BE = 2EC$ and $\angle ADC = \angle BAE$. Find $\angle BAC$.

1999.1 Let $ABCDE$ be a regular pentagon. The star $ACEBD$ has area 1. AC and BE meet at P, while BD and CE meet at Q. Find the area of $APQD$.

2000.3 A rectangular piece of paper has top edge AD. A line L from A to the bottom edge makes an angle x with the line AD. We want to trisect x. We take B and C on the vertical ege through A such that $AB = BC$. We then fold the paper so that C goes to a point C' on the line L and A goes to a point A' on the horizontal line through B. The fold takes B to B'. Show that AA' and AB' are the required trisectors.

2001.1 A sheet of rectangular $ABCD$ paper, of area 1, is folded along its diagonal AC and then unfolded, then it is bent so that vertex A coincides with vertex C and then unfolded, leaving the crease MN, as shown below.
a) Show that the quadrilateral $AMCN$ is a rhombus.
b) If the diagonal AC is twice the width AD, what is the area of the rhombus $AMCN$?
https://2.bp.blogspot.com/-TeQ0QKYGzOQ/Xp9lQcaLbsI/AAAAAAAAL2E/JLXwEIPSr4U79tATcYzmcJjK5b
RqACK4BGAYYCw/s400/2001%2Baomb%2Bl2.png

2001.3 Given a positive integer h, show that there are a finite number of triangles with integer sides a, b, c and altitude relative to side c equal to h .

2001.6 An altitude of a convex quadrilateral is a line through the midpoint of a side perpendicular to the opposite side. Show that the four altitudes are concurrent iff the quadrilateral is cyclic.

2002.1 Let XYZ be a right triangle of area 1 m^2 . Consider the triangle $X'Y'Z'$ such that X' is the symmetric of X wrt side YZ, Y' is the symmetric of Y wrt side XZ and Z' is the symmetric of Z wrt side XY. Calculate the area of the triangle $X'Y'Z'$.

2002.5 Let ABC be a triangle inscribed in a circle of center O and P be a point on the arc AB, that does not contain C. The perpendicular drawn fom P on line BO intersects AB at S and BC at T. The perpendicular drawn from P on line AO intersects AB at Q and AC at R. Prove that:
a) PQS is an isosceles triangle
b) $PQ^2 = QR = ST$

2003.3 The triangle ABC is inscribed in the circle S and $AB < AC$. The line containing A and is perpendicular to BC meets S in P ($P \neq A$). Point X is on the segment AC and the line BX intersects S in Q ($Q \neq B$). Show that $BX = CX$ if, and only if, PQ is a diameter of S.

2003.5 Given a circle and a point A inside the circle, but not at its center. Find points B, C, D on the circle which maximise the area of the quadrilateral $ABCD$.

2004.2 In the figure, ABC and DAE are isosceles triangles ($AB = AC = AD = DE$) and the angles BAC and ADE have measures 36^o.
a) Using geometric properties, calculate the measure of angle $\angle EDC$.
b) Knowing that $BC = 2$, calculate the length of segment DC.
c) Calculate the length of segment AC .
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2004.5 Let D be the midpoint of the hypotenuse AB of a right triangle ABC. Let O_1 and O_2 be the circumcenters of the ADC and DBC triangles, respectively.
a) Prove that $\angle O_1DO_2$ is right.
b) Prove that AB is tangent to the circle of diameter O_1O_2 .

2005.2 In the right triangle ABC, the perpendicular sides AB and BC have lengths 3 cm and 4 cm, respectively. Let M be the midpoint of the side AC and let D be a point, distinct from A, such that $BM = MD$ and $AB = BD$.
a) Prove that BM is perpendicular to AD.
b) Calculate the area of the quadrilateral $ABDC$.

2005.6 The angle B of a triangle ABC is 120^o. Let M be a point on the side AC and K a point on the extension of the side AB, such that BM is the internal bisector of the angle $\angle ABC$ and CK is the external bisector corresponding to the angle $\angle ACB$. The segment MK intersects BC at point P. Prove that $\angle APM = 30^o$.

2006.2 Among the 5-sided polygons, as many vertices as possible collinear , that is, belonging to a single line, is three, as shown below. What is the largest number of collinear vertices a 12-sided polygon can have?
https://cdn.artofproblemsolving.com/attachments/1/1/53d419efa4fc4110730a857ae6988fc923eb1
png
Attention: In addition to drawing a 12-sided polygon with the maximum number of vertices collinear , remember to show that there is no other 12-sided polygon with more vertices collinear than this one.

2006.5 Let ABC be an acute triangle with orthocenter H. Let M, N and R be the midpoints of AB, BC an AH, respectively. If $\angle ABC = 70^\circ$, compute $\angle MNR$.

2007.1 Let ABC be a triangle with circumcenter O. Let P be the intersection of straight lines BO and AC and ω be the circumcircle of triangle AOP. Suppose that $BO = AP$ and that the measure of the arc OP in ω, that does not contain A, is 40^o. Determine the measure of the angle $\angle OBC$.
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s1600/2007%2Bomb%2Bl2.png

2007.5 $\triangle ABC$ is a right isosceles triangle. Choose points K and M from the hypotenuse AB, such that $K \in AM$ and $\angle KCM = 45$°. Prove that $(AK)^2 + (MB)^2 = (KM)^2$

Thanks for any help.

2008.3 Let P be a convex pentagon with all sides equal. Prove that if two of the angles of P add to 180^o, then it is possible to cover the plane with P, without overlaps.

2008.5 Let ABC be an acutangle triangle and O, H its circumcenter, orthocenter, respectively. If $\frac{AB}{\sqrt{2}} = BH = OB$, calculate the angles of the triangle ABC .

2009.2 Let A be one of the two points of intersection of two circles with centers X, Y respectively.The tangents at A to the two circles meet the circles again at B, C. Let a point P be located so that $PXAY$ is a parallelogram. Show that P is also the circumcenter of triangle ABC.

2009.5 An ant walks on the plane as follows: initially, it walks 1 cm in any direction. After, at each step, it changes the trajectory direction by 60^o left or right and walks 1 cm in that direction. It is possible that it returns to the point from which it started in
(a) 2008 steps?
(b) 2009 steps?
https://cdn.artofproblemsolving.com/attachments/8/b/d4c0d03c67432c4e790b465a74a876b93824
png

2009.6 Let ABC be a triangle and O its circumcenter. Lines AB and AC meet the circumcircle of OBC again in $B_1 \neq B$ and $C_1 \neq C$, respectively, lines BA and BC meet the circumcircle of OAC again in $A_2 \neq A$ and $C_2 \neq C$, respectively, and lines CA and CB meet the circumcircle of OAB in $A_3 \neq A$ and $B_3 \neq B$, respectively. Prove that lines A_2A_3, B_1B_3 and C_1C_2 have a common point.

2010.2 Let $ABCD$ be a parallelogram and ω be the circumcircle of the triangle ABD. Let E, F be the intersections of ω with lines BC, CD respectively . Prove that the circumcenter of the triangle CEF lies on ω.

2010.5 The diagonals of an cyclic quadrilateral $ABCD$ intersect at O. The circumcircles of triangle AOB and COD intersect lines BC and AD, for the second time, at points $M, N, P$and Q. Prove that the $MNPQ$ quadrilateral is inscribed in a circle of center O.

2010.6 The three sides and the area of a triangle are integers. What is the smallest value of the area of this triangle?

2011.2 Let $ABCD$ be a convex quadrilateral such that $AD = DC, AC = AB$ and $\angle ADC = \angle CAB$. If M and N are midpoints of the AD and AB sides, prove that the MNC triangle is isosceles.

2011.5 Inside a square of side 16 are placed 1000 points. Show that it is possible to put a equilateral triangle of side $2\sqrt{3}$ in the plane so that it covers at least 16 of these points.

2012.3 Let be a triangle ABC, the midpoint of the AC and N side, and the midpoint of the AB side. Let r and s reflect the straight lines BM and CN on the straight BC, respectively. Also define D and E as the intersection of the lines r and s and the line MN, respectively. Let X and Y be the intersection points between the circumcircles of the triangles BDM and CEN, Z the intersection of the lines BE and CD and W the intersection between the lines r and s. Prove that XY, WZ, and BC are concurrents.

2012.4 The figure below shows a regular $ABCDE$ pentagon inscribed in an equilateral triangle MNP . Determine the measure of the angle CMD.
`http://4.bp.blogspot.com/-LLT7hB7QwiA/Xp9fXOsihLI/AAAAAAAAL14/5lPsjXeKfYwIr5DyRAKRyOTbrX_zx1xHQCK4BGAYYCw/s200/2012%2Bobm%2B12.png`

2013.3 Let ABC a triangle. Let D be a point on the circumcircle of this triangle and let E, F be the feet of the perpendiculars from A on DB, DC, respectively. Finally, let N be the midpoint of EF. Let $M \neq N$ be the midpoint of the side BC . Prove that the lines NA and NM are perpendicular.

2013.5 Let ABC be a scalene triangle and AM is the median relative to side BC. The diameter circumference AM intersects for the second time the side AB and AC at points P and Q, respectively, both different from A. Assuming that PQ is parallel to BC, determine the angle measurement ¡BAC.

Any solution without trigonometry?

2013.6 Consider a positive integer n and two points A and B in a plane. Starting from point A, n rays and starting from point B, n rays are drawn so that all of them are on the same half-plane defined by the line AB and that the angles formed by the $2n$ rays with the segment AB are all acute. Define circles passing through points A, B and each meeting point between the rays. What is the smallest number of **distinct** circles that can be defined by this construction?

2014.2 Let AB be a diameter of the circunference ω, let C and D be point in this circunference, such that CD is perpedicular to AB. Let E be the point of intersection of the segment CD and the segment AB, and a point P that is in the segment CD, P is different of E. The lines AP and BP intersects ω, in F and G respectively. If O is the circumcenter of triangle EFG, show that the area of triangle OCD is invariant, independent of the position of the point P.

2014.4 Let $ABCD$ be a square and O is your center. Let E, F, G, H points in the segments AB, BC, CD, AD respectively, such that $AE = BF = CG = DH$. The line OA intersects the segment EH in the point X, OB intersects EF in the point Y, OC intersects FG in the point Z and OD intersects HG in the point W. If the $(EFGH) = 1$. Find: $(ABCD) \times (XYZW)$
Note (P) denote the area of the polygon P.

2015.2 Let $ABCD$ be a convex quadrilateral. Let E be the intersection of line AB with the line CD, and F is the intersection of line BC with the line AD.
Let P and Q be the foots of the perpendicular of E to the lines AD and BC respectively, and let R and S be the foots of the perpendicular of F to the lines AB and CD, respectively.The point T is the intersection of the line ER with the line FS.
a) Show that, there exists a circle that passes in the points E, F, P, Q, R and S.
b)Show that, the circumcircle of triangle RST is tangent with the circumcircle of triangle QRB.

2015.3 Let ABC be a triangle and n a positive integer. Consider on the side BC the points $A_1, A_2, ..., A_{2^n-1}$ that divide the side into 2^n equal parts, that is, $BA_1 = A_1A_2 = ... = A_{2^n-2}A_{2^n-1} = A_{2^n-1}C$. Set the points $B_1, B_2, ..., B_{2^n-1}$ and $C_1, C_2, ..., C_{2^n-1}$ on the sides CA and AB, respectively, analogously. Draw the line segments $AA_1, AA_2, ..., AA_{2^n-1}$, $BB_1, BB_2, ..., BB_{2^n-1}$ and $CC_1, CC_2, ..., CC_{2^n-1}$. Find, in terms of n, the number of regions into which the triangle is divided.

2015.6 Let ABC a scalene triangle and AD, BE, CF your angle bisectors, with D in the segment BC, E in the segment AC and F in the segment AB. If $\angle AFE = \angle ADC$.
Determine $\angle BCA$.

2016.2 The inner bisections of the $\angle ABC$ and $\angle ACB$ angles of the ABC triangle are at I. The BI parallel line that runs through the point A finds the CI line at the point D. The CI parallel line for A finds the BI line at the point E. The lines BD and CE are at the point F. Show that F, A, and I are collinear if and only if $AB = AC$.

2016.4 Consider a scalene triangle ABC with $AB < AC < BC$. The AB side mediator cuts the B side at the K point and the AC prolongation at the U. point. AC side cuts BC side at O point and AB side extension at G point. Prove that the $GOKU$ quad is cyclic, meaning its four vertices are at same circumference

2017.1 The points X, Y, Z are marked on the sides AB, BC, AC of the triangle ABC, respectively. Points A', B', C' are on the XZ, XY, YZ sides of the triangle XYZ, respectively, so that $\frac{AB}{A'B'} = \frac{AB}{A'B'} = \frac{BC}{B'C'} = 2$ and $ABB'A'$, $BCC'B'$, $ACC'A'$ are trapezoids in which the sides of the triangle ABC are bases.
a) Determine the ratio between the area of the trapezium $ABB'A'$ and the area of the triangle $A'B'X$.
b) Determine the ratio between the area of the triangle XYZ and the area of the triangle ABC.

2017.5 Let ABC be a triangle with $AB \neq AC$, and let K be the incenter. Let P and Q be the other points of the intersections of the circumcircle of triangle BCK with the lines AB and AC, respectively. Let D be intersection of AK and BC.
a) Show that P, Q, D are collinear,
b) Let T, diiferent from P, be the intersection of the circumcircles of triangles PQB and QDC. Prove that T lies on the circumcircle of the triangle ABC.

[full wording with source: 2017 Brazil MO level 2 p5 - OBM - above was only the 1st part]

2018.3 Let ABC be an acute-angled triangle with circumcenter O and orthocenter H. The circle with center X_a passes in the points A and H and is tangent to the circumcircle of ABC. Define X_b, X_c analogously, let O_a, O_b, O_c the symmetric of O to the sides BC, AC and AB, respectively. Prove that the lines O_aX_a, O_bX_b, O_cX_c are concurrents.

2018.4 a) In XYZ triangle, the incircle touches XY and XZ in T and W, respectively. Prove that:

$$XT = XW = \frac{XY + XZ - YZ}{2}$$

Let ABC a triangle and D the foot of the perpendicular of A in BC. Let I, J be the incenters of ABD and ACD, respectively. The incircles of ABD and ACD touch AD in M and N, respectively. Let P be where the incircle of ABC touches AB. The circle with centre A and radius AP intersects AD in K.

b) Show that $\triangle IMK \cong \triangle KNJ$.
c) Show that $IDJK$ is cyclic.

2019.3 Let ABC be an acutangle triangle inscribed in a circle Γ of center O. Let D be the height of the vertex A. Let E and F be points over Γ such that $AE = AD = AF$. Let P and Q be the intersection points of the EF with sides AB and AC respectively. Let X be the second intersection point of Γ with the circle circumscribed to the triangle APQ. Show that the lines XD and AO meet at a point above sobre

2019.4 Let ABC be an acutangle triangle and D any point on the BC side. Let E be the symmetrical of D in AC and F is the symmetrical D relative to AB. A straight ED intersects straight AB at G, while straight FD intersects the line AC in H. Prove that the points A, E, F, G and H are on the same circumference.

2019.6 On the Cartesian plane, all points with both integer coordinates are painted blue. Two blue points are said to be *mutually visible* if the line segment that connects them has no other blue points . Prove that there is a set of 2019 blue points that are mutually visible two by two.

No plano cartesiano, todos os pontos com ambas coordenadas inteiras são pintados de azul. Dois pontos azuis são ditos mutuamente visíveis se o segmento de reta que os conecta não possui outros pontos azuis. Prove que existe um conjunto de 2019 pontos azuis que são mutuamente visíveis dois a dois.

PS. There is a comment about problem being wrong / incorrect here (`https://artofproblemsolving.com/community/c6h1957974p14780265`)

2020.1 Let ABC be an acute triangle and AD a height. The angle bissector of $\angle DAC$ intersects DC at E. Let F be a point on AE such that BF is perpendicular to AE. If $\angle BAE = 45$, find $\angle BFC$.

2020.5 Let ABC be a triangle and M the midpoint of AB. Let circumcircles of triangles CMO and ABC intersect at K where O is the circumcenter of ABC. Let P be the intersection of lines OM and CK. Prove that $\angle PAK = \angle MCB$.

2021.3 Let ABC be a scalene triangle and ω is your incircle. The sides BC, CA and AB are tangents to ω in X, Y, Z respectively. Let M be the midpoint of BC and D is the intersection point of BC with the angle bisector of $\angle BAC$. Prove that $\angle BAX = \angle MAC$ if and only if YZ passes by the midpoint of AD.

2021.5 Let ABC be an acute-angled triangle. Let A_1 be the midpoint of the arc BC which contain the point A. Let A_2 and A_3 be the foot(s) of the perpendicular(s) of the point A_1 to the lines AB and AC, respectively. Define B_2, B_3, C_2, C_3 analogously.
a) Prove that the line A_2A_3 cuts BC in the midpoint.
b) Prove that the lines A_2A_3, B_2B_3 and C_2C_3 are concurrents.

2021.7 Let ABC be a triangle with $\angle ABC = 90°$. The square $BDEF$ is inscribed in $\triangle ABC$, such that D, E, F are in the sides AB, CA, BC respectively. The inradius of $\triangle EFC$ and $\triangle EDA$ are c and b, respectively. Four circles $\omega_1, \omega_2, \omega_3, \omega_4$ are drawn inside the square $BDEF$, such that the radius of ω_1 and ω_3 are both equal to b and the radius of ω_2 and ω_4 are both equal to a. The circle ω_1 is tangent to ED, the circle ω_3 is tangent to BF, ω_2 is tangent to EF and ω_4 is tangent to BD, each one of these circles are tangent to the two closest circles and the circles ω_1 and ω_3 are tangents. Determine the ratio $\frac{c}{a}$.
