Estonia Team Selection Test 2011

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–	Day 1
1	Two circles lie completely outside each other.Let A be the point of intersection of internal common tangents of the circles and let K be the projection of this point onto one of their external common tangents.The tangents,different from the common tangent,to the circles through point K meet the circles at M_1 and M_2.Prove that the line AK bisects angle M_1KM_2.
2	Let n be a positive integer. Prove that for each factor m of the number $1+2+\cdots+n$ such that $m \geq n$, the set $\{1, 2, \ldots, n\}$ can be partitioned into disjoint subsets, the sum of the elements of each being equal to m. **Edit**:Typographical error fixed.
3	Does there exist an operation $*$ on the set of all integers such that the following conditions hold simultaneously: (1) for all integers x, y, z, $(x*y)*z = x*(y*z)$; (2) for all integers x and y, $x*x*y = y*x*x = y$?
–	Day 2
4	Let a, b, c be positive real numbers such that $2a^2+b^2=9c^2$.Prove that $\dfrac{2c}{a}+\dfrac{c}{b} \geq \sqrt{3}$.
5	Prove that if n and k are positive integers such that $1<k<n-1$,Then the binomial coefficient $\binom{n}{k}$ is divisible by at least two different primes.
6	On a square board with m rows and n columns, where $m \leq n$, some squares are colored black in such a way that no two rows are alike. Find tha biggest integer k such that, for every possible coloring to start with, one can always color k columns entirely red in such a way that still no two rows are alike.