Czech-Polish-Slovak Junior Match 2012

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by parmenides51

–	Individual
1	Point P lies inside the triangle ABC. Points K, L, M are symmetrics of point P wrt the midpoints of the sides BC, CA, AB. Prove that the straight AK, BL, CM intersect at one point.
2	Determine all three primes (a, b, c) that satisfied the equality $a^2 + ab + b^2 = c^2 + 3$.
3	Different points A, B, C, D lie on a circle with a center at the point O at such way that $\angle AOB = \angle BOC = \angle COD = 60^o$. Point P lies on the shorter arc BC of this circle. Points K, L, M are projections of P on lines AO, BO, CO respectively . Show that (a) the triangle KLM is equilateral, (b) the area of triangle KLM does not depend on the choice of the position of point P on the shorter arc BC
4	Prove that among any 51 vertices of the 101-regular polygon there are three that are the vertices of an isosceles triangle.
5	Positive integers a, b, c satisfying the equality $a^2 + b^2 = c^2$. Show that the number $\frac{1}{2}(c - a)(c - b)$ is square of an integer.
–	Team
1	There are a lot of different real numbers written on the board. It turned out that for each two numbers written, their product was also written. What is the largest possible number of numbers written on the board?
2	On the circle k, the points A, B are given, while AB is not the diameter of the circle k. Point C moves along the long arc AB of circle k so that the triangle ABC is acute. Let D, E be the feet of the altitudes from A, B respectively. Let F be the projection of point D on line AC and G be the projection of point E on line BC. (a) Prove that the lines AB and FG are parallel. (b) Determine the set of midpoints S of segment FG while along all allowable positions of point C.
3	Prove that if n is a positive integer then $2(n^2 + 1) - n$ is not a square of an integer.

4 A rhombus $ABCD$ is given with $\angle BAD = 60^o$. Point P lies inside the rhombus such that $BP = 1$, $DP = 2$, $CP = 3$. Determine the length of the segment AP.

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5 Find all triplets (a, k, m) of positive integers that satisfy the equation $k + a^k = m + 2a^m$.

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6 The 8×8 board is covered with the same shape as in the picture to the right (each of the shapes can be rotated 90^o) so that any two do not overlap or extend beyond the edge of the chessboard. Determine the largest possible number of fields of this chessboard can be covered as described above.
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