Czech-Polish-Slovak Junior Match 2013

www.artofproblemsolving.com/community/c1093554

by parmenides51

– Individual

1 Determine all pairs (x, y) of integers for which satisfy the equality $\sqrt{x-\sqrt{y}}+\sqrt{x+\sqrt{y}}=\sqrt{xy}$

2 Each positive integer should be colored red or green in such a way that the following two conditions are met:
- Let n be any red number. The sum of any n (not necessarily different) red numbers is red.
- Let m be any green number. The sum of any m (not necessarily different) green numbers is green.
Determine all such colorings.

3 The $ABCDE$ pentagon is inscribed in a circle and $AB = BC = CD$. Segments AC and BE intersect at K, and Segments AD and CE intersect at point$L$. Prove that $AK = KL$.

4 Determine the largest two-digit number d with the following property:
for any six-digit number $\overline{aabbcc}$ number d is a divisor of the number $\overline{aabbcc}$ if and only if the number d is a divisor of the corresponding three-digit number $\overline{abc}$.

Note The numbers $a \neq 0, b$ and c need not be different.

5 Point M is the midpoint of the side AB of an acute triangle ABC. Point P lies on the segment AB, and points S_1 and S_2 are the centers of the circumcircles of APC and BPC, respectively. Show that the midpoint of segment S_1S_2 lies on the perpendicular bisector of segment CM.

– Team

1 Decide whether there are infinitely many primes p having a multiple in the form $n^2 + n + 1$ for some natural number n

2 Find all natural numbers n such that the sum of the three largest divisors of n is 1457.

3 In a certain group there are $n \geq 5$ people, with every two people who do not know each other exactly having one mutual friend and no one knows everyone else. Prove 5 of n people, may sit at a circle around the table so that each of them sits between
a) friends,
b) strangers.

4 Let $ABCD$ be a convex quadrilateral with $\angle DAB = \angle ABC = \angle BCD > 90^o$. The circle circumscribed around the triangle ABC intersects the sides AD and CD at points K and L, respectively, different from any vertex of the quadrilateral $ABCD$. Segments AL and CK intersect at point P. Prove that $\angle ADB = \angle PDC$.

5 Let a, b, c be positive real numbers for which $ab + ac + bc \ge a + b + c$. Prove that $a + b + c \ge 3$.

6 There is a square $ABCD$ in the plane with $|AB| = a$. Determine the smallest possible radius value of three equal circles to cover a given square.