V Caucasus Mathematical Olympiad

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–	Juniors
–	First day
1	By one magic nut, Wicked Witch can either turn a flea into a beetle or a spider into a bug; while by one magic acorn, she can either turn a flea into a spider or a beetle into a bug. In the evening Wicked Witch had spent 20 magic nuts and 23 magic acorns. By these actions, the number of beetles increased by 5. Determine what was the change in the number of spiders. (Find all possible answers and prove that the other answers are impossible.)
2	Let ω_1 and ω_2 be two non-intersecting circles. Let one of its internal tangents touches ω_1 and ω_2 at A_1 and A_2, respectively, and let one of its external tangents touches ω_1 and ω_2 at B_1 and B_2, respectively. Prove that if $A_1B_2 = A_2B_1$, then $A_1B_2 \perp A_2B_1$.
3	Let a_n be a sequence given by $a_1 = 18$, and $a_n = a_{n-1}^2 + 6a_{n-1}$, for $n > 1$. Prove that this sequence contains no perfect powers.
4	Positive integers $n, k > 1$ are given. Pasha and Vova play a game on a board $n \times k$. Pasha begins, and further they alternate the following moves. On each move a player should place a border of length 1 between two adjacent cells. The player loses if after his move there is no way from the bottom left cell to the top right without crossing any order. Determine who of the players has a winning strategy.
–	Second day
5	Find the number of pairs of positive integers a and b such that $a \leq 100\,000$, $b \leq 100\,000$, and $$\frac{a^3 - b}{a^3 + b} = \frac{b^2 - a^2}{b^2 + a^2}.$$
6	All vertices of a regular 100-gon are colored in 10 colors. Prove that there exist 4 vertices of the given 100-gon which are the vertices of a rectangle and which are colored in at most 2 colors.
7	A regular triangle ABC is given. Points K and N lie in the segment AB, a point L lies in the segment AC, and a point M lies in the segment BC so that $CL = AK$, $CM = BN$, $ML = KN$. Prove that $KL \parallel MN$.

8 Let real a, b, and c satisfy
$$abc + a + b + c = ab + bc + ca + 5.$$
Find the least possible value of $a^2 + b^2 + c^2$.

– Seniors

– First day

1 Determine if there exists a finite set A of positive integers satisfying the following condition: for each $a \in A$ at least one of two numbers $2a$ and $\frac{a}{3}$ belongs to A.

2 Let ω_1 and ω_2 be two non-intersecting circles. Let one of its internal tangents touches ω_1 and ω_2 at A_1 and A_2, respectively, and let one of its external tangents touches ω_1 and ω_2 at B_1 and B_2, respectively. Prove that if $A_1B_2 \perp A_2B_1$, then $A_1B_2 = A_2B_1$.

3 Peter and Basil play the following game on a horizontal table 1×2019. Initially Peter chooses n positive integers and writes them on a board. After that Basil puts a coin in one of the cells. Then at each move, Peter announces a number s among the numbers written on the board, and Basil needs to shift the coin by s cells, if it is possible: either to the left, or to the right, by his decision. In case it is not possible to shift the coin by s cells neither to the left, nor to the right, the coin stays in the current cell. Find the least n such that Peter can play so that the coin will visit all the cells, regardless of the way Basil plays.

4 Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers m and n the number $f(m)+n-m$ is divisible by $f(n)$.

– Second day

5 See Juniors 6

6 Morteza wishes to take two real numbers S and P, and then to arrange six pairwise distinct real numbers on a circle so that for each three consecutive numbers at least one of the two following conditions holds:
1) their sum equals S
2) their product equals P.
Determine if Morteza's wish could be fulfilled.

7 In $\triangle ABC$ with $AB \neq AC$ let M be the midpoint of AB, let K be the midpoint of the arc BAC in the circumcircle of $\triangle ABC$, and let the perpendicular bisector of AC meet the bisector of $\angle BAC$ at P . Prove that A, M, K, P are concyclic.

8 Peter wrote 100 distinct integers on a board. Basil needs to fill the cells of a table 100×100 with integers so that the sum in each rectangle 1×3 (either vertical, or horizontal) is equal to one of the numbers written on the board. Find the greatest n such that, regardless of numbers written by Peter, Basil can fill the table so that it would contain each of numbers $(1, 2, ..., n)$ at least once (and possibly some other integers).