Turkey Team Selection Test 2020

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– Day 1

1 Find all pairs of (a,b) positive integers satisfying the equation:

$$\frac{a^3+b^3}{ab+4}=2020$$

2 $A_1A_2A_3A_4$ is a tangential quadrilateral with perimeter p_1 and sum of the diagonals k_1 .$B_1B_2B_3B_4$ is a tangential quadrilateral with perimeter p_2 and sum of the diagonals k_2 .Prove that $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ are congruent squares if

$$p_1^2+p_2^2=(k_1+k_2)^2$$

3 66 dwarfs have a total of 111 hats. Each of the hats belongs to a dwarf and colored by 66 different colors. Festivities are organized where each of these dwarfs wears their own hat. There is no dwarf pair wearing the same colored hat in any of the festivities. For any two of the festivities, there exist a dwarf wearing a hat of a different color in these festivities. Find the maximum value of the number of festivities that can be organized.

– Day 2

4 Let Z^+ be positive integers set. $f:\mathbb{Z}^+\to\mathbb{Z}^+$ is a function and we show $f\circ f\circ ...\circ f$ with f_l for all $l\in\mathbb{Z}^+$ where f is repeated l times. Find all $f:\mathbb{Z}^+\to\mathbb{Z}^+$ functions such that

$$(n-1)^{2020}<\prod_{l=1}^{2020}f_l(n)<n^{2020}+n^{2019}$$

for all $n\in\mathbb{Z}^+$

5 There is at least one friend pair in a class of students with different names. Students in an ordered list of some of the students write the names of all their friends who are not currently written on the blackboard, in order. If each student on the list wrote at least one name on the board and the name of each student with at least one friend on the blackboard at the end of the process, call this list a $golden\ list$. Prove that there exists a $golden\ list$ such that number of students in this list is even.

6 In a triangle $\triangle ABC$, D and E are respectively on AB and AC such that $DE \parallel BC$. P is the intersection of BE and CD. M is the second intersection of (APD) and (BCD) , N is the second intersection of (APE) and (BCE). w is the circle passing through M and N and tangent to BC. Prove that the lines tangent to w at M and N intersect on AP.

– Day 3

7 $A_1, A_2, B_1, B_2, C_1, C_2$ are points on a circle such that $A_1A_2 \parallel B_1B_2 \parallel C_1C_2$. M is a point on same circle MA_1 and B_2C_2 intersect at X , MB_1 and A_2C_2 intersect at Y, MC_1 and A_2B_2 intersect at Z .Prove that X, Y, Z are collinear.

8 Let x, y, z be real numbers such that $0 < x, y, z < 1$. Find the minimum value of:

$$\frac{xyz(x + y + z) + (xy + yz + zx)(1 - xyz)}{xyz\sqrt{1 - xyz}}$$

9 For a, n positive integers we show number of different integer 10-tuples $(x_1, x_2, ..., x_{10})$ on $(modn)$ satistfying $x_1x_2...x_{10} = a(modn)$ with $f(a, n)$. Let a, b given positive integers ,
a) Prove that there exist a positive integer c such that for all $n \in \mathbb{Z}^+$

$$\frac{f(a, cn)}{f(b, cn)}$$

is constant
b) Find all (a, b) pairs such that minumum possible value of c is 27 where c satisfying condition in (a)