Purple Comet Problems 2018

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– Middle School

1 Find n such that the mean of $\frac{7}{4}$, $\frac{6}{5}$, and $\frac{1}{n}$ is 1.

2 The following figure is made up of many 2×4 tiles such that adjacent tiles always share an edge of length 2. Find the perimeter of this figure.
`https://cdn.artofproblemsolving.com/attachments/0/6/fcdf06eda94901b9bbe7a2857e1c6f05c5061`
`png`

3 The fraction

$$\left(\frac{\frac{1}{3}+1}{3}+\frac{1+\frac{1}{3}}{3}\right) / \left(\frac{3}{\frac{1}{3+1}+\frac{1}{1+3}}\right)$$

can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

4 The diagram below shows a large square with each of its sides divided into four equal segments. The shaded square whose sides are diagonals drawn to these division points has area 13. Find the area of the large square.
`https://cdn.artofproblemsolving.com/attachments/8/3/bee223ef39dea493d967e7ebd557581695403`
`png`

5 The positive integer m is a multiple of 101, and the positive integer n is a multiple of 63. Their sum is 2018. Find $m-n$.

6 Find the greatest integer n such that 10^n divides

$$\frac{2^{10^5}5^{2^{10}}}{10^{5^2}}$$

7 Bradley is driving at a constant speed. When he passes his school, he notices that in 20 minutes he will be exactly $\frac{1}{4}$ of the way to his destination, and in 45 minutes he will be exactly $\frac{1}{3}$ of the way to his destination. Find the number of minutes it takes Bradley to reach his destination from the point where he passes his school.

8 On side AE of regular pentagon $ABCDE$ there is an equilateral triangle AEF, and on side AB of the pentagon there is a square $ABHG$ as shown. Find the degree measure of angle AFG.

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png

9 For some $k > 0$ the lines $50x + ky = 1240$ and $ky = 8x + 544$ intersect at right angles at the point (m, n). Find $m + n$.

10 The triangle below is divided into nine stripes of equal width each parallel to the base of the triangle. The darkened stripes have a total area of 135. Find the total area of the light colored stripes.
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png

11 Find the number of positive integers less than 2018 that are divisible by 6 but are not divisible by at least one of the numbers 4 or 9.

12 Line segment $\overline{AB}$ has perpendicular bisector $\overline{CD}$, where C is the midpoint of $\overline{AB}$. The segments have lengths $AB = 72$ and $CD = 60$. Let R be the set of points P that are midpoints of line segments $\overline{XY}$, where X lies on $\overline{AB}$ and Y lies on $\overline{CD}$. Find the area of the region R.

13 Suppose x and y are nonzero real numbers simultaneously satisfying the equations $x + \frac{2018}{y} = 1000$ and $\frac{9}{x} + y = 1$.
Find the maximum possible value of $x + 1000y$.

14 Find the number of ordered quadruples of positive integers (a, b, c, d) such that $ab + cd = 10$.

15 There are integers $a_1, a_2, a_3, ..., a_{240}$ such that $x(x + 1)(x + 2)(x + 3)...(x + 239) = \sum_{n=1}^{240} a_n x^n$.
Find the number of integers k with $1 \leq k \leq 240$ such that ak is a multiple of 3.

16 On $\triangle\ ABC$ let D be a point on side $\overline{AB}$, F be a point on side $\overline{AC}$, and E be a point inside the triangle so that $\overline{DE} \parallel \overline{AC}$ and $\overline{EF} \parallel \overline{AB}$. Given that $AF = 6, AC = 33, AD = 7, AB = 26$, and the area of quadrilateral $ADEF$ is 14, nd the area of $\triangle\ ABC$.

17 Let a, b, c, and d be real numbers such that $a^2 + b^2 + c^2 + d^2 = 3a + 8b + 24c + 37d = 2018$. Evaluate $3b + 8c + 24d + 37a$.

18 Rectangle $ABCD$ has side lengths $AB = 6\sqrt{3}$ and $BC = 8\sqrt{3}$. The probability that a randomly chosen point inside the rectangle is closer to the diagonal $\overline{AC}$ than to the outside of the rectangle is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

19 Two identical blue blocks, two identical red blocks, two identical green blocks, and two identical purple blocks are placed next to each other in a row. Find the number of distinct arrangements of these blocks where no blue block is placed next to a red block, and no green block is placed next to a purple block.

20 Let $ABCD$ be a square with side length 6. Circles X, Y, and Z are congruent circles with centers inside the square such that X is tangent to both sides $\overline{AB}$ and $\overline{AD}$, Y is tangent to both sides $\overline{AB}$ and $\overline{BC}$, and Z is tangent to side $\overline{CD}$ and both circles X and Y. The radius of the circle X can be written $m-\sqrt{n}$, where m and n are positive integers. Find $m+n$.

– High School

1 Find the positive integer n such that $\frac{1}{2}\cdot\frac{3}{4}+\frac{5}{6}\cdot\frac{7}{8}+\frac{9}{10}\cdot\frac{11}{12}=\frac{n}{1200}$.

2 A triangle with side lengths 16, 18, and 21 has a circle with radius 6 centered at each vertex. Find n so that the total area inside the three circles but outside of the triangle is $n\pi$.
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3 Find x so that the arithmetic mean of $x, 3x, 1000$, and 3000 is 2018.

4 The following diagram shows a grid of 36 cells. Find the number of rectangles pictured in the diagram that contain at least three cells of the grid.
https://cdn.artofproblemsolving.com/attachments/a/4/e9ba3a35204ec68c17a364ebf92cc107eb4d7 png

5 One afternoon at the park there were twice as many dogs as there were people, and there were twice as many people as there were snakes. The sum of the number of eyes plus the number of legs on all of these dogs, people, and snakes was 510. Find the number of dogs that were at the park.

6 Triangle ABC has $AB=AC$. Point D is on side $\overline{BC}$ so that $AD=CD$ and $\angle BAD=36^o$. Find the degree measure of $\angle BAC$.

7 In 10 years the product of Melanie's age and Phil's age will be 400 more than it is now. Find what the sum of Melanie's age and Phil's age will be 6 years from now.

8 Let a and b be positive integers such that $2a-9b+18ab=2018$. Find $b-a$.

9 trapezoid has side lengths $10, 10, 10$, and 22. Each side of the trapezoid is the diameter of a semicircle with the two semicircles on the two parallel sides of the trapezoid facing outside the trapezoid and the other two semicircles facing inside the trapezoid as shown. The region bounded by these four semicircles has area $m+n\pi$, where m and n are positive integers. Find $m+n$.

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10 Find the remainder when 11^{2018} is divided by 100.

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11 Find the number of positive integers $k \le 2018$ for which there exist integers m and n so that $k = 2^m + 2^n$.
For example, $64 = 2^5 + 2^5$, $65 = 2^0 + 2^6$, and $66 = 2^1 + 2^6$.

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12 A jeweler can get an alloy that is 40% gold for 200 dollars per ounce, an alloy that is 60% gold for 300 dollar per ounce, and an alloy that is 90% gold for 400 dollars per ounce. The jeweler will purchase some of these gold alloy products, melt them down, and combine them to get an alloy that is 50% gold. Find the minimum number of dollars the jeweler will need to spend for each ounce of the alloy she makes.

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13 Five lighthouses are located, in order, at points A, B, C, D, and E along the shore of a circular lake with a diameter of 10 miles. Segments AD and BE are diameters of the circle. At night, when sitting at A, the lights from B, C, D, and E appear to be equally spaced along the horizon. The perimeter in miles of pentagon $ABCDE$ can be written $m + \sqrt{n}$, where m and n are positive integers. Find $m + n$.

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14 A complex number z whose real and imaginary parts are integers satis fies $(Re(z))^4 + \left(Re(z^2)\right)^2 + |z|^4 = (2018)(81)$, where $Re(w)$ and $Im(w)$ are the real and imaginary parts of w, respectively. Find $(Im(z))^2$
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15 Let a and b be real numbers such that $\frac{1}{a^2} + \frac{3}{b^2} = 2018a$ and $\frac{3}{a^2} + \frac{1}{b^2} = 290b$. Then $\frac{ab}{b-a} = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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16 If you roll four standard, fair six-sided dice, the top faces of the dice can show just one value (for example, 3333), two values (for example, 2666), three values (for example, 5215), or four values (for example, 4236). The mean number of values that show is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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17 One afternoon a bakery finds that it has 300 cups of flour and 300 cups of sugar on hand. Annie and Sam decide to use this to make and sell some batches of cookies and some cakes. Each batch of cookies will require 1 cup of flour and 3 cups of sugar. Each cake will require 2 cups of flour and 1 cup of sugar. Annie thinks that each batch of cookies should sell for 2 dollars and each cake for 1 dollar, but Sam thinks that each batch of cookies should sell for 1 dollar and each cake should sell for 3 dollars. Find the difference between the maximum dollars of income they can receive if they use Sam's selling plan and the maximum dollars of income they can receive if they use Annie's selling plan.

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18 Find the positive integer k such that the roots of $x^3 - 15x^2 + kx - 1105$ are three distinct collinear points in the complex plane.

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19 Suppose that a and b are positive real numbers such that $3\log_{101}\left(\frac{1{,}030{,}301-a-b}{3ab}\right) = 3 - 2\log_{101}(ab)$. Find $101 - \sqrt[3]{a} - \sqrt[3]{b}$.

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20 Aileen plays badminton where she and her opponent stand on opposite sides of a net and attempt to bat a birdie back and forth over the net. A player wins a point if their opponent fails to bat the birdie over the net. When Aileen is the server (the first player to try to hit the birdie over the net), she wins a point with probability $\frac{9}{10}$. Each time Aileen successfully bats the birdie over the net, her opponent, independent of all previous hits, returns the birdie with probability $\frac{3}{4}$. Each time Aileen bats the birdie, independent of all previous hits, she returns the birdie with probability $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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21 Let x be in the interval $\left(0, \frac{\pi}{2}\right)$ such that $\sin x - \cos x = \frac{1}{2}$. Then $\sin^3 x + \cos^3 x = \frac{m\sqrt{p}}{n}$, where $m, n,$ and p are relatively prime positive integers, and p is not divisible by the square of any prime. Find $m + n + p$.

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22 Positive integers a and b satisfy $a^3 + 32b + 2c = 2018$ and $b^3 + 32a + 2c = 1115$. Find $a^2 + b^2 + c^2$.

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23 Let a, b, and c be integers simultaneously satisfying the equations $4abc + a + b + c = 2018$ and $ab + bc + ca = -507$. Find $|a| + |b| + |c|$.

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24 Five girls and five boys randomly sit in ten seats that are equally spaced around a circle. The probability that there is at least one diameter of the circle with two girls sitting on opposite ends of the diameter is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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25 If a and b are in the interval $\left(0, \frac{\pi}{2}\right)$ such that $13(\sin a + \sin b) + 43(\cos a + \cos b) = 2\sqrt{2018}$, then $\tan a + \tan b = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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26 Let a, b, and c be real numbers. Let $u = a^2 + b^2 + c^2$ and $v = 2ab + 2bc + 2ca$. Suppose $2018u = 1001v + 1024$. Find the maximum possible value of $35a - 28b - 3c$.

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27 Suppose $p < q < r < s$ are prime numbers such that $pqrs + 1 = 4^{p+q}$. Find $r + s$.

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28 In $\triangle ABC$ points D, E, and F lie on side $\overline{BC}$ such that $\overline{AD}$ is an angle bisector of $\angle BAC$, $\overline{AE}$ is a median, and $\overline{AF}$ is an altitude. Given that $AB = 154$ and $AC = 128$, and $9 \times DE = EF$, fi nd the side length BC.

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29 Find the three-digit positive integer n for which $\binom{n}{3}\binom{n}{4}\binom{n}{5}\binom{n}{6}$ is a perfect square.

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30 One right pyramid has a base that is a regular hexagon with side length 1, and the height of the pyramid is 8. Two other right pyramids have bases that are regular hexagons with side length 4, and the heights of those pyramids are both 7. The three pyramids sit on a plane so that their bases are adjacent to each other and meet at a single common vertex. A sphere with radius 4 rests above the plane supported by these three pyramids. The distance that the center of the sphere is from the plane can be written as $\frac{p\sqrt{q}}{r}$, where p, q, and r are relatively prime positive integers, and q is not divisible by the square of any prime. Find $p + q + r$.
