Purple Comet Problems 2019

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by parmenides51

–	Middle School
1	The diagram shows a polygon made by removing six 2×2 squares from the sides of an 8×12 rectangle. Find the perimeter of this polygon. `https://cdn.artofproblemsolving.com/attachments/6/3/c23510c821c159d31aff0e6688edebc81e273` `png`
2	Evaluate $1 + 2 - 3 - 4 + 5 + 6 - 7 - 8 + ... + 2018 - 2019$.
3	The diagram below shows a shaded region bounded by two concentric circles where the outer circle has twice the radius of the inner circle. The total boundary of the shaded region has length 36π. Find n such that the area of the shaded region is $n\pi$. `https://cdn.artofproblemsolving.com/attachments/4/5/c9ffdc41c633cc61127ef585a45ee5e6c0f88` `png`
4	Of the students attending a school athletic event, 80% of the boys were dressed in the school colors, 60% of the girls were dressed in the school colors, and 45% of the students were girls. Find the percentage of students attending the event who were wearing the school colors.
5	The diagram below shows four congruent squares and some of their diagonals. Let T be the number of triangles and R be the number of rectangles that appear in the diagram. Find $T + R$. `https://cdn.artofproblemsolving.com/attachments/1/5/f756bbe67c09c19e811011cb6b18d0ff44be8` `png`
6	Find the value of n such that $\frac{2019+n}{2019-n} = 5$
7	The diagram shows some squares whose sides intersect other squares at the midpoints of their sides. The shaded region has total area 7. Find the area of the largest square. `https://cdn.artofproblemsolving.com/attachments/3/a/c3317eefe9b0193ca15f36599be3f6c22bb09` `png`
8	In the subtraction PURPLE $-$ COMET $=$ MEET each distinct letter represents a distinct decimal digit, and no leading digit is 0. Find the greatest possible number represented by PURPLE.
9	A semicircle has diameter $\overline{AD}$ with $AD = 30$. Points B and C lie on $\overline{AD}$, and points E and F lie on the arc of the semicircle. The two right triangles $\triangle BCF$ and $\triangle CDE$ are congruent. The area of $\triangle BCF$ is $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the

square of any prime. Find $m+n$.
https://cdn.artofproblemsolving.com/attachments/b/c/c10258e2e15cab74abafbac5ff50b1d0fd42e
png

10 Let N be the greatest positive integer that can be expressed using all seven Roman numerals I, V, X, L, C, D, and M exactly once each, and let n be the least positive integer that can be expressed using these numerals exactly once each. Find $N-n$. Note that the arrangement CM is never used in a number along with the numeral D.

11 Find the number of positive integers less than or equal to 2019 that are no more than 10 away from a perfect square.

12 Find the number of ordered triples of positive integers (a,b,c), where a,b,c is a strictly increasing arithmetic progression, $a+b+c=2019$, and there is a triangle with side lengths a, b, and c.

13 Squares $ABCD$ and $AEFG$ each with side length 12 overlap so that $\triangle AED$ is an equilateral triangle as shown. The area of the region that is in the interior of both squares which is shaded in the diagram is $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m+n$.
https://cdn.artofproblemsolving.com/attachments/c/2/a2f8d2a090a6342610c43b3fed8a87fa5d7f0
png

14 For real numbers a and b, let $f(x) = ax+b$ and $g(x) = x^2 - x$. Suppose that $g(f(2)) = 2, g(f(3)) = 0$, and $g(f(4)) = 6$. Find $g(f(5))$.

15 Let a, b, c, and d be prime numbers with $a \le b \le c \le d > 0$.
Suppose $a^2 + 2b^2 + c^2 + 2d^2 = 2(ab + bc - cd + da)$. Find $4a + 3b + 2c + d$.

16 Four congruent semicircular half-disks are arranged inside a circle with radius 4 so that each semicircle is internally tangent to the circle, and the diameters of the semicircles form a 2×2 square centered at the center of the circle as shown. The radius of each semicircular half-disk is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.
https://cdn.artofproblemsolving.com/attachments/f/e/8c0b9fdd69f6b54d39708da94ef2b2d039cb1
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17 Find the greatest integer n such that 5^n divides $2019! - 2018! + 2017!$.

18 Suppose that a, b, c, and d are real numbers simultaneously satisfying $a + b - c - d = 3$ $ab - 3bc + cd - 3da = 4$ $3ab - bc + 3cd - da = 5$
Find $11(a-c)^2 + 17(b-d)^2$.

19 Rectangle $ABCD$ has sides $AB = 10$ and $AD = 7$. Point G lies in the interior of $ABCD$ a distance 2 from side $\overline{CD}$ and a distance 2 from side $\overline{BC}$. Points H, I, J, and K are located on sides $\overline{BC}, \overline{AB}, \overline{AD}$, and $\overline{CD}$, respectively, so that the path $GHIJKG$ is as short as possible. Then $AJ = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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20 Harold has 3 red checkers and 3 black checkers. Find the number of distinct ways that Harold can place these checkers in stacks. Two ways of stacking checkers are the same if each stack of the rst way matches a corresponding stack in the second way in both size and color arrangement. So, for example, the 3 stack arrangement RBR, BR, B is distinct from RBR, RB, B, but the 4 stack arrangement RB, BR, B, R is the same as B, BR, R, RB.

– High School

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1 Ivan, Stefan, and Katia divided 150 pieces of candy among themselves so that Stefan and Katia each got twice as many pieces as Ivan received. Find the number of pieces of candy Ivan received.

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2 The large square in the diagram below with sides of length 8 is divided into 16 congruent squares. Find the
area of the shaded region.
`https://cdn.artofproblemsolving.com/attachments/6/e/cf828197aa2585f5eab2320a43b806160721`
`png`

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3 The mean of $\frac{1}{2}, \frac{3}{4}$, and $\frac{5}{6}$ differs from the mean of $\frac{7}{8}$ and $\frac{9}{10}$ by $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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4 The diagram below shows a sequence of equally spaced parallel lines with a triangle whose vertices lie on these lines. The segment $\overline{CD}$ is 6 units longer than the segment $\overline{AB}$. Find the length of segment $\overline{EF}$.
`https://cdn.artofproblemsolving.com/attachments/8/0/abac87d63d366bf4c4e913fdb1022798379a`
`png`

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5 Evaluate

$$\frac{(2+2)^2}{2^2} \cdot \frac{(3+3+3+3)^3}{(3+3+3)^3} \cdot \frac{(6+6+6+6+6+6)^6}{(6+6+6+6)^6}$$

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6 A pentagon has four interior angles each equal to 110^o. Find the degree measure of the fifth interior angle.

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7 Find the number of real numbers x that satisfy the equation $(3^x)^{x+2} + (4^x)^{x+2} - (6^x)^{x+2} = 1$

8 The diagram below shows a 12 by 20 rectangle split into four strips of equal widths all surrounding an isosceles triangle. Find the area of the shaded region.
https://cdn.artofproblemsolving.com/attachments/9/e/ed6be5110d923965c64887a2ca8e858c9777C
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9 Find the positive integer n such that 32 is the product of the real number solutions of $x^{\log_2(x^3)-n} = 13$

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10 Find the number of positive integers less than 2019 that are neither multiples of 3 nor have any digits that are multiples of 3.

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11 Let $m > n$ be positive integers such that $3(3mn-2)^2 - 2(3m-3n)^2 = 2019$. Find $3m+n$.

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12 The following diagram shows four adjacent 2×2 squares labeled $1, 2, 3$, and 4. A line passing through the lower left vertex of square 1 divides the combined areas of squares $1, 3$, and 4 in half so that the shaded region has area 6. The difference between the areas of the shaded region within square 4 and the shaded region within square 1 is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.
https://cdn.artofproblemsolving.com/attachments/7/4/b9554ccd782af15c680824a1fbef278a4f736
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13 There are relatively prime positive integers m and n so that the parabola with equation $y = 4x^2$ is tangent to the parabola with equation $x = y^2 + \frac{m}{n}$. Find $m+n$.

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14 The circle centered at point A with radius 19 and the circle centered at point B with radius 32 are both internally tangent to a circle centered at point C with radius 100 such that point C lies on segment $\overline{AB}$. Point M is on the circle centered at A and point N is on the circle centered at B such that line MN is a common internal tangent of those two circles. Find the distance MN.
https://cdn.artofproblemsolving.com/attachments/3/d/1933ce259c229d49e21b9a2dcadddea2a6b4C
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15 Suppose a is a real number such that $\sin(\pi \cdot \cos a) = \cos(\pi \cdot \sin a)$. Evaluate $35\sin^2(2a) + 84\cos^2(4a)$.

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16 Find the number of ordered triples of sets (T_1, T_2, T_3) such that
1. each of T_1, T_2, and T_3 is a subset of $\{1, 2, 3, 4\}$,
2. $T_1 \subseteq T_2 \cup T_3$,
3. $T_2 \subseteq T_1 \cup T_3$, and
4. $T_3 \subseteq T_1 \cup T_2$.

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17 The following diagram shows equilateral triangle $\triangle ABC$ and three other triangles congruent to it. The other three triangles are obtained by sliding copies of $\triangle ABC$ a distance $\frac{1}{8}AB$ along

a side of $\triangle\ ABC$ in the directions from A to B, from B to C, and from C to A. The shaded region inside all four of the triangles has area 300. Find the area of $\triangle\ ABC$.
`https://cdn.artofproblemsolving.com/attachments/3/a/8d724563c7411547d3161076015b247e88212`
`png`

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18 A container contains five red balls. On each turn, one of the balls is selected at random, painted blue, and returned to the container. The expected number of turns it will take before all fi ve balls are colored blue is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

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19 Find the remainder when $\prod_{n=3}^{33} 2n^4 - 25n^3 + 33n^2$ is divided by 2019.

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20 In the diagram below, points D, E, and F are on the inside of equilateral $\triangle\ ABC$ such that D is on $\overline{AE}$, E is on $\overline{CF}$, F is on $\overline{BD}$, and the triangles $\triangle\ AEC$, $\triangle\ BDA$, and $\triangle\ CFB$ are congruent. Given that $AB = 10$ and $DE = 6$, the perimeter of $\triangle\ BDA$ is $\frac{a+b\sqrt{c}}{d}$, where a, b, c, and d are positive integers, b and d are relatively prime, and c is not divisible by the square of any prime. Find $a+b+c+d$.
`https://cdn.artofproblemsolving.com/attachments/8/6/98da82fc1c26fa13883a47ba6d45a015622b2`
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21 Each of the 48 faces of eight $1\times1\times1$ cubes is randomly painted either blue or green. The probability that these eight cubes can then be assembled into a $2\times2\times2$ cube in a way so that its surface is solid green can be written $\frac{p^m}{q^n}$, where p and q are prime numbers and m and n are positive integers. Find $p+q+m+n$.

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22 Let a and b positive real numbers such that $(65a^2+2ab+b^2)(a^2+8ab+65b^2) = (8a^2+39ab+7b^2)^2$. Then one possible value of $\frac{a}{b}$ satis es $2\left(\frac{a}{b}\right) = m+\sqrt{n}$, where m and n are positive integers. Find $m+n$.

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23 Find the number of ordered pairs of integers (x, y) such that

$$\frac{x^2}{y} - \frac{y^2}{x} = 3\left(2+\frac{1}{xy}\right)$$

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24 A 12-sided polygon is inscribed in a circle with radius r. The polygon has six sides of length $6\sqrt{3}$ that alternate with six sides of length 2. Find r^2.

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25 The letters $AAABBCC$ are arranged in random order. The probability no two adjacent letters will be the same is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

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26 Let D be a regular dodecahedron, which is a polyhedron with 20 vertices, 30 edges, and 12 regular pentagon faces. A tetrahedron is a polyhedron with 4 vertices, 6 edges, and 4 triangular faces. Find the number of tetrahedra with positive volume whose vertices are vertices of D.

https://cdn.artofproblemsolving.com/attachments/c/d/44d11fa3326780941d0b6756fb2e5989c2dc5
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27 Binhao has a fair coin. He writes the number $+1$ on a blackboard. Then he flips the coin. If it comes up heads (H), he writes $+\frac{1}{2}$, and otherwise, if he flips tails (T), he writes $-\frac{1}{2}$. Then he flips the coin again. If it comes up heads, he writes $+\frac{1}{4}$, and otherwise he writes $-\frac{1}{4}$. Binhao continues to flip the coin, and on the nth flip, if he flips heads, he writes $+\frac{1}{2n}$, and otherwise he writes $-\frac{1}{2n}$. For example, if Binhao flips HHTHTHT, he writes $1+\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}-\frac{1}{128}$. The probability that Binhao will generate a series whose sum is greater than $\frac{1}{7}$ is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+10q$.

28 There are positive integers m and n such that $m^2-n=32$ and $\sqrt[5]{m+\sqrt{n}}+\sqrt[5]{m-\sqrt{n}}$ is a real root of the polynomial $x^5-10x^3+20x-40$. Find $m+n$.

29 In a right circular cone, A is the vertex, B is the center of the base, and C is a point on the circumference of the base with $BC=1$ and $AB=4$. There is a trapezoid $ABCD$ with $\overline{AB}\parallel\overline{CD}$. A right circular cylinder whose surface contains the points A, C, and D intersects the cone such that its axis of symmetry is perpendicular to the plane of the trapezoid, and $\overline{CD}$ is a diameter of the cylinder. A sphere radius r lies inside the cone and inside the cylinder. The greatest possible value of r is $\frac{a\sqrt{b}-c}{d}$, where a, b, c, and d are positive integers, a and d are relatively prime, and b is not divisible by the square of any prime. Find $a+b+c+d$.

30 A *derangement* of the letters $ABCDEF$ is a permutation of these letters so that no letter ends up in the position it began such as $BDECFA$. An *inversion* in a permutation is a pair of letters xy where x appears before y in the original order of the letters, but y appears before x in the permutation. For example, the derangement $BDECFA$ has seven inversions: AB, AC, AD, AE, AF, CD, and CE. Find the total number of inversions that appear in all the derangements of $ABCDEF$.