Problems from the 2019-2020 Winter SDPC. Middle School division does 1,2,3,5,6,7, High School division does 2,3,4,6,7,8.

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by mira74

–	Session 1
1	Six people sit at a circular table (in the shape of a regular hexagon) such that no two friends sit next to or across from each other. Find, with proof, the maximum number of unordered pairs of people that can be friends.
2	Let $S = \{a_0, a_1, a_2, a_3, \dots\}$ be a set of positive integers with $1 = a_0 < a_1 < a_2 < a_3 < \dots$. For a subset T of S, let $\sigma(T)$ be the sum of the elements of T. For instance, $\sigma(\{1,2,3\}) = 6$. By convention, $\sigma(\emptyset) = 0$, where $\emptyset$ denotes an empty set. Call a number n representable if there exists a subset T of S such that $\sigma(T) = n$. We aim to prove for any set S satisfying $a_{k+1} \le 2a_k$ for every $k \ge 0$, that all non-negative integers are representable. (a) Prove there is a unique value of a_1, and find this value. Use this to determine, with proof, all possible sets $\{a_0, a_1, a_2, a_3\}$. (Hint: there are 7 possible sets.) [Not for credit] I recommend that you show that for all 7 sets in part (a), every integer between 0 and $a_3 - 1$ is representable. (Note that this does not depend on the values of $a_4, a_5, a_6, \dots$.) (b) Show that if $a_k \le n \le a_{k+1} - 1$, then $0 \le n - a_k \le a_k - 1$. (c) Prove that any non-negative integer is representable.
3	Find, with proof, all functions f mapping integers to integers with the property that for all integers m, n, $$f(m) + f(n) = \max\left(f(m+n), f(m-n)\right).$$
4	Farmer John ties his goat to a number of ropes of varying lengths in the Euclidean plane. If he ties the goat to k ropes centered at Q_1, Q_2, ... Q_k with lengths ℓ_1, ℓ_2, ... ℓ_k (respectively), the goat can reach any point R such that $\ell_j \ge RQ_j$ for all $j \in \{1, 2, 3, \dots k\}$. Suppose that Farmer John has planted grass at a finite set of points P_1, P_2, ... P_n, and sets the ropes such that the goat can reach all of these points. What is, in terms of the points, the largest possible lower bound on the area of the region that the goat can reach?
–	Session 2

5 Let $a_1, a_2, \ldots$ be a sequence of real numbers such that $a_1 = 4$ and $a_2 = 7$ such that for all integers n, $\frac{1}{a_{2n-1}}, \frac{1}{a_{2n}}, \frac{1}{a_{2n+1}}$ forms an arithmetic progression, and $a_{2n}, a_{2n+1}, a_{2n+2}$ forms an arithmetic progression. Find, with proof, the prime factorization of a_{2019}.

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6 Fix a positive integer n. Let $a_1, a_2, \ldots$ be a sequence of positive integers such that for all $1 \le j \le n$, $a_j = j$, and for all $j > n$, a_j is the largest value of $\min(a_i, a_{j-i})$ among $i = 1, 2, \ldots j-1$. For example, if $n = 3$, we have $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, and $a_4 = 2$ since $\min(a_1, a_3) = 1$, $\min(a_2, a_2) = 2$, and $\min(a_3, a_1) = 1$. We will determine the values of a_k for sufficiently large k.

(a) Show that $a_i \in \{1, 2, 3, \ldots n\}$ for all i.

(b) Show that if $a_x \ge n-1$ and $a_y \ge n-1$, $a_{x+y} \ge n-1$.

(c) Show that for some positive integer N, $a_k \in \{n-1, n\}$ for all $k \ge N$.

(d) Show that $a_k = n$ if and only if $n \mid k$.

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7 Let a, b be positive integers. Find, with proof, the maximum possible value of $a\lceil b\lambda \rceil - b\lfloor a\lambda \rfloor$ for irrational λ.

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8 Let ABC be a triangle with circumcircle Γ. If the internal angle bisector of $\angle A$ meets BC and Γ at D and E respectively. Let O_1 be the center of the circle through A and D tangent to BC, let the external angle bisector of $\angle A$ meet Γ at F, and let FO_1 meet Γ at some point $P \neq F$. Show that the circumcircle of DEP is tangent to BC.
