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– Session 1

1 Show that there exists some **positive** integer k with

$$\gcd(2012, 2020) = \gcd(2012 + k, 2020)$$

$$= \gcd(2012, 2020 + k) = \gcd(2012 + k, 2020 + k).$$

2 Consider a function $f : \mathbb{Z} \to \mathbb{Z}$. We call an integer a *spanning* if for all integers $b \neq a$, there exists a positive integer k with $f^k(a) = b$. Find, with proof, the maximum possible number of *spanning* numbers of f.

Note: $\mathbb{Z}$ represents the set of all integers, so f is a function from the set of integers to itself. $f^k(a)$ is defined as f applied k times to a.

3 Find all polynomials P with integer coefficients such that for all positive integers x, y,

$$\frac{P(x) - P(y)}{x^2 + y^2}$$

evaluates to an integer (in particular, it can be zero).

4 Let $\triangle ABC$ be an acute, scalene triangle with orthocenter H, and let AH meet the circumcircle of $\triangle ABC$ at a point $D \neq A$. Points E and F are chosen on AC and AB such that $DE \perp AC$ and $DF \perp AB$. Show that BE, CF, and the line through H parallel to EF concur.

– Session 2

5 Is there a function f from the positive integers to themselves such that

$$f(a)f(b) \geq f(ab)f(1)$$

with equality **if and only if** $(a-1)(b-1)(a-b) = 0$?

6 Let $ABCD$ be an isosceles trapezoid inscribed in circle ω, such that $AD \| BC$. Point E is chosen on the arc BC of ω not containing A. Let BC and DE intersect at F. Show that if E is chosen such that $EB = EC$, the area of AEF is maximized.

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7 Find all pairs of positive integers a, b with

$$a^a + b^b \mid (ab)^{|a-b|} - 1.$$

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8 Find all angles $0 < \theta < 90^\circ$ for which there exists an angle $0 < \beta < 90^\circ$ such that a right triangle with angles $90^\circ, \theta, 90^\circ - \theta$ can be tiled by a finite number of isosceles triangles with angles $\beta, \beta, 180^\circ - 2\beta$. (The isosceles triangles are not necessarily pairwise congruent, but they are pairwise similar.)

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