National Mathematical Olympiad 2014

www.artofproblemsolving.com/community/c1109932

by parmenides51, Konigsberg, fattypiggy123

–	2nd Round
1	The quadrilateral ABCD is inscribed in a circle which has diameter BD. Points A and B are symmetric to A and B with respect to the line BD and AC respectively. If the lines AC, BD intersect at P and AC, BD intersect at Q, prove that PQ is perpendicular to AC.
2	Find all functions from the reals to the reals satisfying $$f(xf(y)+x)=xy+f(x)$$
3	Let $0<a_1<a_2<\cdots<a_n$ be real numbers. Prove that $$\left(\frac{1}{1+a_1}+\frac{1}{1+a_2}+\cdots+\frac{1}{1+a_n}\right)^2\le\frac{1}{a_1}+\frac{1}{a_2-a_1}+\cdots+\frac{1}{a_n-a_{n-1}}.$$
4	Fill up each square of a 50 by 50 grid with an integer. Let G be the configuration of 8 squares obtained by taking a 3 by 3 grid and removing the central square. Given that for any such G in the 50 by 50 grid, the sum of integers in its squares is positive, show there exist a 2 by 2 square such that the sum of its entries is also positive.
5	Determine the largest odd positive integer n such that every odd integer k with $1<k<n$ and $\gcd(k,n)=1$ is a prime.